Formalizing $\pi_4(\mathbb{S}^3)$ and computing a Brunerie number in Cubical Agda

Anders Mörtberg (j.w.w. Axel Ljungström)

Workshop in Honour of Thierry Coquand's 60th Birthday, Göteborg, Aug 26, 2022

The fourth homotopy group of the 3-sphere in HoTT

Guillaume Brunerie's PhD thesis contains a synthetic proof in Book HoTT of:

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The fourth homotopy group of the 3-sphere is $\mathbb{Z}/2\mathbb{Z}$, that is, $\pi_4(\mathbb{S}^3) \simeq \mathbb{Z}/2\mathbb{Z}$

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Furthermore, the proof is fully constructive!

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The theorem can hence be phrased as: *"there exists a number $\beta : \mathbb{Z}$ such that $\pi_4(\mathbb{S}^3) \simeq \mathbb{Z}/\beta\mathbb{Z}$"*

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On p. 85 Brunerie says (for $n := |\beta|$):

> *This result is quite remarkable in that even though it is a constructive proof, it is not at all obvious how to actually compute this n. At the time of writing, we still haven't managed to extract its value from its definition. A complete and concise definition of this number n is presented in appendix B, for the benefit of someone wanting to implement it in a prospective proof assistant. In the rest of this thesis, we give a mathematical proof in homotopy type theory that n = 2.*

The Cubical paradigm in HoTT/UF

As we saw in Steve's talk Thierry worked hard on giving constructive meaning to HoTT/UF during the IAS special year (2012-2013)

Breakthrough: Bezem-Coquand-Huber (BCH, 2014) constructive model of univalence

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Breakthrough: Bezem-Coquand-Huber (BCH, 2014) constructive model of univalence

Led to *lots* of developments:

- Cohen-Coquand-Huber-M. (CCHM) model and cubical type theory
- Huber: canonicity for CCHM cubical type theory
- Cartesian cubical models and type theories (Awodey, Angiuli-Favonia-Harper, Angiuli-Brunerie-Coquand-Harper-Favonia-Licata)
- ...

Computing the Brunerie number

This enabled us to implement a variety of cubical proof assistants: `cubical`, `cubicaltt`, `yacctt`, **RedPRL**, **redtt**, **cooltt**, `Cubical Agda`...

As these satisfy canonicity it should *in principle* be possible to use them to compute the Brunerie number...

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Computing the Brunerie number, a (probably incomplete) history

- 2013: Guillaume presents informal definition of the Brunerie number at an IAS seminar
- December 2014: Guillaume visits Chalmers and tries to compute it with Thierry Coquand and Simon Huber using `cubical` (based on BCH model)
- Spring 2015: I join forces with them and spend a lot of time trying to benchmark and optimize the Haskell implementation of `cubical`
- 2016: Guillaume finishes thesis with definition in Appendix B (based on `cubical` code)

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- 2016: Guillaume finishes thesis with definition in Appendix B (based on `cubical` code)
- Spring/summer 2017: I port the proof to `cubicaltt` (based on CCHM), but computation runs out of memory (on Inria server with 64GB RAM)
- June 2017: another attempt in `cubicaltt` with the MRC group in Snowbird (Vikraman Choudhury, Paul Gustafson, Dan Licata, Ian Orton, and Jon Sterling). Optimizes the definition of the number, without luck
- Late 2017: I visit Guillaume repeatedly at the IAS and simplify the definition a lot, computation goes slightly further but still runs out of memory

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- 2018: various attempts to run parts of the computation in various cartesian cubical systems (`yacctt` and **redtt**) as well as in `Cubical Agda`, no luck
- June 2018: Favonia tries running the `cubicaltt` computation on a super computer with 1TB of ram, computation stopped after $\sim$ 90 hours(?)
- Summer 2018: Dagstuhl meeting where the cubical group (Jon Sterling, Carlo Angiuli, Favonia, Dan Licata, Simon Huber, Ian Orton, Guillaume Brunerie) found various new optimizations to cubical evaluation ("Dagstuhl lemma"), did not help with computation

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- 2022: Breakthrough with Axel Ljungström!

2022 breakthroughs

In 2020 Axel wrote a master thesis on "Computing Cohomology with Cubical Agda" supervised by me and Guillaume. He then started a PhD with the aim of formalizing Guillaume's proof that $\pi_4(\mathbb{S}^3) \simeq \mathbb{Z}/2\mathbb{Z}$ in `Cubical Agda`...

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- April 7: with some tricks we get one of these numbers to normalize to -2 in just a few seconds in `Cubical Agda`!
- May 23: the formalization that this normalized number is really a Brunerie number is finished

Outline

1. Formalizing Brunerie's proof
2. Ljungström's new proof and the simplified Brunerie number
3. Conclusions and future work

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Contents

Brunerie's theorem: part 1 (chapters 1–3)

In the first half of the thesis (chapters 1–3) Guillaume constructs a map $g : \mathbb{S}^3 \to \mathbb{S}^2$

g is defined as the composition of a sequence of (pointed) maps $\mathbb{S}^3 \to \mathbb{S}^1 * \mathbb{S}^1 \to \mathbb{S}^2 \vee \mathbb{S}^2 \to \mathbb{S}^2$

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Let $e : \pi_3(\mathbb{S}^2) \simeq \mathbb{Z}$ and define $\beta := e(|\ g\ |)$, the first main theorem is then that:

Theorem (Brunerie, Corollary 3.4.5)

We have $\pi_4(\mathbb{S}^3) \simeq \mathbb{Z}/\beta\mathbb{Z}$

Brunerie's proof: part 1 (chapters 1–3)

The proof of this theorem uses:

- Hopf fibration
- LES of homotopy groups of a fibration
- Freudenthal suspension theorem
- James construction[1]
- The Blakers-Massey theorem
- Whitehead products

This is quite complicated synthetic HoTT, but all of it was formalizable and the proofs didn't contain any major surprises (except for a typo in the definition of Whitehead products)

[1] General form actually not needed, can do a direct encode-decode proof instead.

We first prove the following more general version which isn't more complicated to prove.

Proposition 3.3.2. *Given two types A and B, there is a map $W_{A,B} : A * B \to \Sigma A \vee \Sigma B$ such that*

$$\Sigma A \times \Sigma B \simeq 1 \sqcup^{A*B} (\Sigma A \vee \Sigma B)$$

and such that the induced map $\Sigma A \vee \Sigma B \to \Sigma A \times \Sigma B$ is $i^{\vee}_{\Sigma A, \Sigma B}$.

Proof. We use the 3×3-lemma with the diagram

$$\begin{tikzcd}
\Sigma A & B \arrow[l, "\mathsf{north}"'] \arrow[r] & 1 \\
B \arrow[u, "\mathsf{south}"] \arrow[d, "\mathsf{id}"'] & A \times B \arrow[l, "\mathsf{snd}"] \arrow[r, "\mathsf{fst}"'] \arrow[u, "\mathsf{snd}"'] \arrow[d, "\mathsf{snd}"] \arrow[ul, phantom, "\Swarrow\alpha"] & A \arrow[u] \arrow[d] \\
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where $\alpha : A \times B \to \mathsf{north} =_{\Sigma A} \mathsf{south}$ is defined by $\alpha(x, y) := \mathsf{merid}(x)$.

The pushout of the top row is equivalent to $\Sigma A \vee \Sigma B$, the pushout of the middle row is equivalent to the join $A * B$ and the pushout of the bottom row is contractible, so the pushout of the pushouts of the rows is equivalent to $1 \sqcup^{A*B} (\Sigma A \vee \Sigma B)$ for the map $A * B \to \Sigma A \vee \Sigma B$ defined by

$$\begin{aligned}
W_{A,B} &: A * B \to \Sigma A \vee \Sigma B, \\
W_{A,B}(\mathsf{inl}(a)) &:= \mathsf{inr}(\mathsf{north}), \\
W_{A,B}(\mathsf{inr}(b)) &:= \mathsf{inl}(\mathsf{north}), \\
\mathsf{ap}_{W_{A,B}}(\mathsf{push}(a, b)) &:= \mathsf{ap}_{\mathsf{inr}}(\varphi_B(b)) \cdot \mathsf{push}(\star_{\mathbf{1}}) \cdot \mathsf{ap}_{\mathsf{inl}}(\varphi_A(a)).
\end{aligned}$$

The pushouts of the left and of the right columns are both equivalent to ΣA, and the pushout of the middle column is equivalent to $\Sigma A \times B$. Moreover, the horizontal map on the left between $\Sigma A \times B$ and ΣA is equal to fst, as can be proved by induction using the definition of α. The horizontal map on the right is also equal to fst. Hence the pushout of the pushout of the columns is equivalent to $\Sigma A \times \Sigma B$. Therefore we have

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Brunerie's proof: part 2 (chapters 4–6)

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- Symmetric monoidal structure of smash products

$$
\begin{tikzcd}
& ((A \otimes B) \otimes C) \otimes D \arrow[dl, "\alpha_{A,B,C} \otimes \mathsf{id}_D"'] \arrow[dr, "\alpha_{A \otimes B,C,D}"] & \\
(A \otimes (B \otimes C)) \otimes D \arrow[d, "\alpha_{A,B \otimes C,D}"'] & & (A \otimes B) \otimes (C \otimes D) \arrow[d, "\alpha_{A,B,C \otimes D}"] \\
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- This gives graded ring structure of the *cup product* $\smile : H^i(X) \times H^j(X) \to H^{i+j}(X)$

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- The *Mayer-Vietoris sequence*:

$$\begin{tikzcd}
\widetilde{H}^{n+1}(D) \arrow[r, "i"] & \widetilde{H}^{n+1}(A) \times H^{n+1}(B) \arrow[r, "\Delta"] & H^{n+1}(C) \arrow[dll, "d"'] \\
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- The *Gysin sequence*:

$$\mathbb{S}^{n-1} \longrightarrow E \xrightarrow{p} B$$

$$\ldots \longrightarrow H^{i-1}(E) \longrightarrow H^{i-n}(B) \xrightarrow{\smile e} H^{i}(B) \xrightarrow{p^*} H^{i}(E) \longrightarrow \ldots$$

Brunerie's proof: part 2 (chapters 4–6)

- The *Hopf Invariant* homomorphism:

Definition 5.4.1. Given a pointed map $f : \mathbb{S}^{2n-1} \to \mathbb{S}^n$, we define

$$\begin{aligned} C_f &:= \mathbf{1} \sqcup^{\mathbb{S}^{2n-1}} \mathbb{S}^n, \\ \alpha_f &:= (i^*)^{-1}(\mathbf{c}_n) : H^n(C_f), \\ \beta_f &:= p^*(\mathbf{c}_{2n}) : H^{2n}(C_f), \end{aligned}$$

Definition 5.4.2. The *Hopf invariant* of a pointed map $f : \mathbb{S}^{2n-1} \to \mathbb{S}^n$ is the integer $H(f) : \mathbb{Z}$ such that

$$\alpha_f^2 = H(f)\beta_f,$$

where α_f^2 is $\alpha_f \smile \alpha_f$.

Brunerie's proof: part 2 (chapters 4–6)

- The *Iterated Hopf Construction*:

$$\begin{tikzcd}
A \arrow[d, "\mathsf{id}"'] & A \times (A \sqcup^{A\times A} A) \arrow[l, "\mathsf{fst}"'] \arrow[r, "(a,x)\mapsto\nu'_a(x)"] \arrow[d, "(a,x)\mapsto(a,\nu'_a(x))"] & \sum_{x:\Sigma A} H(x) \arrow[d, "\mathsf{id}"] \\
A & A \times \sum_{x:\Sigma A} H(x) \arrow[l, "\mathsf{fst}"] \arrow[r, "\mathsf{snd}"'] & \sum_{x:\Sigma A} H(x)
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  $\implies$ The graded ring structure of the cup product
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We call this a 1-*coherent* symmetric monoidal structure because we do not ask the fillers of the diagrams to satisfy any further coherence condition. It's an open question to give a definition in homotopy type theory of the notion of fully coherent (or even only n-coherent) symmetric monoidal structure, but here we only need the 1-coherent structure of the smash product. The following result is the main result of this section even though we essentially admit it.

Proposition 4.1.2. *The smash product is a* 1*-coherent symmetric monoidal product on pointed types.*

Sketch of proof. Putting the unit aside for a moment, we have to define six functions of the form

$$(x : A \wedge B) \to P(x),$$

four of the form

$$(x : (A \wedge B) \wedge C) \to P(x),$$

two of the form

$$(x : A \wedge (B \wedge C)) \to P(x),$$

and one of the form

$$(x : ((A \wedge B) \wedge C) \wedge D) \to P(x),$$

where each time $P(x)$ is either a smash product like $B \wedge A$ or $A \wedge (B \wedge C)$, or an equality in a smash product between combinations of some of those functions.

The idea is that the smash product $A \wedge B$ can be seen as the product $A \times B$ where all elements of the form $(a, \star_B)$ and $(\star_A, b)$ have been identified together. Therefore, in order to define a map out of $A \wedge B$ it should be enough to define it on elements of the form $\mathsf{proj}(a, b)$ in such a way that the image of elements of the form $\mathsf{proj}(a, \star_B)$ and $\mathsf{proj}(\star_A, b)$

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Symmetric monoidal structure of smash products

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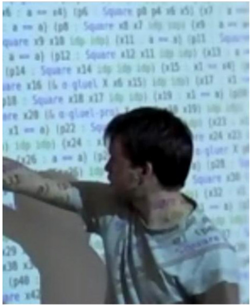
Guillaume tried very hard to do it using Agda-metaprogramming in Agda:



But did not succeed with everything: pentagon missing, hexagon takes 7 minutes and 8GB of RAM to typecheck...

Workaround: alternative definition of the cup product

Recall that there is an adjunction:

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This way we completely formalize the construction of the (integral) cohomology ring $H^*(X;\mathbb{Z})$, for details see *Synthetic Integral Cohomology in Cubical Agda* (Brunerie-Ljungström-M., CSL 2022)

Graded commutativity is most difficult and takes $\sim$ 900LOC, however Tim Baumann's HoTT Agda proof is $\sim$ 5000LOC

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Having filled the hole in Guillaume's proof we managed to formalize the rest of the thesis! But, still no progress on computing the Brunerie number...

Outline

1. Formalizing Brunerie's proof
2. Ljungström's new proof and the simplified Brunerie number
3. Conclusions and future work

New proof

Having finished the formalization Axel realized that one can actually simplify the proof a lot and completely avoid the second half of Brunerie's thesis

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Idea: trace the maps by hand using clever tricks and choices

Details: `https://homotopytypetheory.org/2022/06/09/the-brunerie-number-is-2/`

Sketch of new proof

Recall that $\beta := e(|g|)$ for $e : \pi_3(\mathbb{S}^2) \simeq \mathbb{Z}$ and $g : \mathbb{S}^3 \to \mathbb{S}^2$. The goal is to show that $|\beta| = 2$

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One of Axel's tricks in the proof is to define $\pi_3^*(A) := ||\mathbb{S}^1 * \mathbb{S}^1 \to_* A||_0$ and work with it instead so that f can be avoided

Sketch of new proof

We can now decompose $e : \pi_3(\mathbb{S}^2) \simeq \mathbb{Z}$ as:

$$\pi_3(\mathbb{S}^2) \overset{e_1}{\simeq} \pi_3^*(\mathbb{S}^2) \overset{e_2}{\simeq} \pi_3^*(\mathbb{S}^1 * \mathbb{S}^1) \overset{e_3}{\simeq} \pi_3^*(\mathbb{S}^3) \overset{e_4}{\simeq} \mathbb{Z}$$

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We can also give explicit definitions of

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such that

$$e_1(|g|) = |g_1| \qquad e_2(|g_1|) = |g_2| \qquad e_3(|g_2|) = |g_3| \qquad e_4(|g_3|) = -2$$

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The first 3 equalities are not definitional and requires some clever choices, but (surprisingly) a variation of the last one holds by refl in `Cubical Agda`!

File Edit Options Buffers Tools Agda Help

Save Undo

```
-- We also have a much more direct proof in Cubical.Homotopy.Group.Pi4S3.DirectProof,
-- not relying on any of the more advanced constructions in chapters
-- 4-6 in Brunerie's thesis (but still using chapters 1-3 for the
-- construction). For details see the header of that file.

π₄S³≃ℤ/2ℤ-direct : GroupEquiv (π 4 𝕊³) (ℤGroup/ 2)
π₄S³≃ℤ/2ℤ-direct = DirectProof.BrunerieGroupEquiv

-- This direct proof allows us to define a much simplified version of
-- the Brunerie number:
β' : ℤ
β' = fst DirectProof.computer η₃'

-- This number computes definitionally to -2 in a few seconds!
β'≡-2 : β' ≡ -2
β'≡-2 = refl

-- Combining all of this gives us the desired equivalence of groups by
-- computation as conjectured in Brunerie's thesis:
π₄S³≃ℤ/2ℤ-computation : GroupEquiv (π 4 𝕊³) (ℤGroup/ 2)
π₄S³≃ℤ/2ℤ-computation = DirectProof.BrunerieGroupEquiv''
```

U:--- **Summary.agda** Bot (112,0) Git:inducedstruct (Agda:Checked +2)

U:%*- *All Done* All (1,0) (AgdaInfo)

Outline

Conclusions

We have 3 new and fully formalized synthetic proofs that $\pi_4(\mathbb{S}^3) \simeq \mathbb{Z}/2\mathbb{Z}$:

1. Streamlined and complete proof following Guillaume's thesis
2. Axel's new direct elementary proof which avoids part 2 of the thesis completely
3. The new computational proof which involves normalizing one of these Brunerie numbers

The first two proofs are expressable in Book HoTT, while the third crucially relies on normalization of terms involving univalence and HITs (so expressable in cubical systems, and maybe H.O.T.T.)

Future work

- Why does only $e_4(|g_3|)$ terminate? What about the other numbers?
- The computation is not very stable, composition with refl in certain places can make it run seemingly forever... Why?!
- Does the computation terminate in other cubical systems or is there something special about `Cubical Agda`?
- Which optimizations to `Cubical Agda` were actually necessary to get the computation to terminate?
- Can we compute other interesting numbers and invariants? Cohomology provides a rich source of examples, as does proofs that various groups are finitely generated...

Congratulations Thierry!

Questions?