A comparison of topological toposes

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Collab with Thierry

PhD on constructive functional analysis. TC in committee
Seven joint papers on constructive maths
Esp. constructive functional analysis
Using ideas of constructive algebra, FT, topos th, Bishop's CM
$\rightarrow$ Basis for new field on quantum topos theory
EU-STREP Formath project
Formalizing maths at scale
$\rightarrow$ Basis for high assurance cryptography
HoTT

many happy visits to Chalmers, and an immense amount of very informative emails!

Formalizing Blockchains

Currently, the biggest industrial support for type theory.
Centers in INRIA(Coq),Edinburgh(agda),CMU(lean),Aarhus(Coq)
My group is working on:
Verified extraction (rust, functional languages, ...)
Program verification of functional smart contracts
Distributed systems
Cryptographic primitives (industry grade)
Cryptographic protocols
(modular Easycrypt like system, but in Coq)

Three toposes

Now: exploration of intuitionistic and homotopic ideas.
Warning: metatheory is classical

Topos= categorical model of HOL/CST/ETT
Generalization of topological spaces
Geometric morphism: continuous map between toposes
Preserves geometric logic $\wedge, \vee, \exists$. But not $\rightarrow, \forall$
Obs: the geometric realization: $sSet \rightarrow Top$
maps points, lines, triangles, ... to their topological counterparts
has good categorical properties
Johnstone: this should be a geometric morphism
But Top is not a topos
First try: Giraud big topos of sheaves over topological spaces
Solution: Topos J of continuous M-Sets over the monoid $End(N_\infty)$

Classifying theories

Every Grothendieck topos classifies a geometric theory
E.g. Sh(X) classifies the geometric theory of points of X
Each model x of X in $\mathcal{E}$ gives a unique morphism $Sh(X) \to \mathcal{E}$.
Fact: sSets classifies the theory of (classical) linear orders.
J models LLPO
For decidable P, Q over $\mathbb{N}$:

$$\neg(\exists x, P(x) \wedge \exists y, Q(y)) \Rightarrow \neg(\exists x, P(x)) \vee \neg(\exists y, Q(y))$$

From LLPO: the Dedekind interval is a classical linear order
So, there is a unique geometric morphism: $sSet \to J$
Fact: this coincides with the geometric realization
It lands in Seq as a good subcategory of J

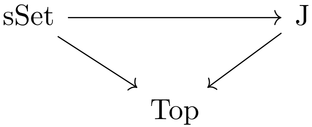
Extends to cSets

Can we extend the homotopical structure?

sSets is a (combinatorial) model category.
Quillen model categories aim formalize abstract homotopy
Homotopy hypothesis: ‘this model is faithful’:
geometric realization is part of the Quillen equivalence between
the classical model structure on Top and
the classical model structure on sSet

Can we extend the homotopical structure?

How about J?



Thm: We can (right) transfer the QMS to J
We do not yet know whether this gives a Quillen equivalence
Thm: $y : Top \to J$ is a right Quillen functor

Can we extend the homotopical structure?

Hard to build a model of Id on Top (vdBG)
Hard to build a model of cubical type theory on Top (OP18)

Problem: strictness of composition
Idea: no shrinking. paths of abitrary length
Moore Paths: maps $R^+ \to X$ as opposed to $I \to X$
Pitts-Orton (PO19) build a more complicated ('topological') topos to model their internal Moore Paths
This gives a model of MLTT+funext with non-trivial Id-types.
Similar model of Moore paths on J (since it satisfies LLPO)

What is a topological topos?

* A topos extending a subcategory of Top (such as J)?
* A sheaf topos over a topological site

Topological site: subcategory of Top[1] closed under open subspaces

Covers: a cover by open subspaces

Fourman/Moerdijk/...

[1] Can also be done with formal spaces

Comparison

Johnstone: they are similar, but is not specific
Fourman (Continuous truth): Topological/localic site
Model for Brouwer's intuitionistic mathematics

Escardo/Xu: EX=continuous M-Sets, where M is End(C)
Prop: Indeed a topological site, by Comparison Lemma
So, all the intuitionistic theorems hold

Both contain Lim as concrete sheaves
C-Space=Lim only classically

Comparison, internal logic

J: LLPO, Dedekind=Cauchy (Shulman), DC?
EX: Fan theorem, Continuity for C, DC, Continuous choice
Bar induction classically (Moerdijk/Reyes)

Thm: $J \vdash N \cong 2^{2^N}$
Note: LLPO contradicts continuity

Pyk

Pyknotic Sets[2] (Barwick/Haine)
Condensed sets (Scholtze/Clause)

Pyknotic=thick, dense, compact

aim: convenient framework for homological algebra

Comp: category of compact Hausdorff spaces with finite covers
Comp is a pretopos

(exact and extensive/quotients, images, good sums)

Prop: Pyk is the topos completion

[2] we ignore size issues: κ, Shulman small sheaves

Pyk

$y(I)$ satisfies *analytical* LLPO

$$\forall x \in I, x \leqslant 0 \vee x \geqslant 0$$

Clausen: So, target of geometric realization
Yoneda: $y : Comp \to Pyk$ extends to
an embedding $y : Top \to Pyk$
y has a left-adjoint: reflection into Comp gen spaces (Scholtze)

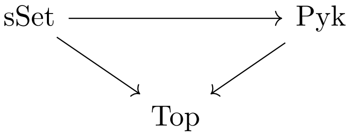


diagram of left adjoint functors
sSet and *Top* carry QMS and are part of a Q-equivalence
Q: Does Pyk carry QMS?
Prop: (Like for J): The Kan model structure transfers along the underlying adjunction
Q: Quillen equivalence?

Pyk

Likely hard to model HoTT, just like for Top
Modelling CTT is not possible using OP/LOPS:
coFib needs to be stable (but is not)

Pyk and EX

Prop: Embedding End(C) to Comp extends to
a geometric morphism EX $\rightarrow$ Pyk
Synthetic fan: $N \rightarrow 2^{2^N}$ is an isomorphism
Prop: Synthetic fan holds in Pyk
Proof: This holds in Comp, and y preserves exponentials

Open Q: what is the relation between I_C, yI and I_D in Pyk?

We have analytical LLPO for yI

Similarities

All three contain the Lim, category of Kuratowski limit spaces
Interesting quasi-subtoposes (concrete sheaves):
Spanier's quasitopological (Pyk), C-spaces (Ex), Limit spaces (J)
(q-topos: roughly, a topos without unique choice)

Contain the Kleene-Kreisel functionals
(interpreted as internal exponentials of N)

In EX and J: Cauchy=Dedekind

Coquand

On a model of choice sequences

Presents a variant of the EX topos, but finite products of localizations of Cantor space are presented as Boolean algebras. (In the spirit of constructive algebra)
Same presheaves as condensed sets, but:
The pro-etale topology used for condensed sets is much finer than the big Zariski topology (open covers)
Gives rise to a geometric morphism

Differences

Coquand considers Boolean Zariski topos, similar to EX
These are sheaves over a topological site
Classify the theory of Boolean algebras such that

$$\forall a, a = 0 \vee a = 1$$

Since, Zariski topos classify local rings

Pyk is the pretopos completion of Comp
y preserves coherent logic
Pyk classifies the geometric theory of the site Comp
(Johnstone/Makkai and Reyes. Not very informative here)

J and Pyk

The base sites of J and Pyk are "test spaces"
J: converging sequences
Condensed sets: converging ultrafilters

The functors from Top to J and Pyk are embeddings when restricted to subcategories of topological spaces uniquely determined by maps from the respective test spaces

J: sequential topological spaces
Pyk: Pyk these are the compactly generated spaces

Future work

Better understanding of the internal language
Use of the internal language
Pontryagin duality?
Duality between compact and discrete groups (somewhat like fan)

Conclusions

Comparison of three three topological toposes
J and Pyk are extensions of topological spaces which model LLPO and are targets for geometric realization and carry Moore Paths

Toposes over topological sites model intuitiontistic maths
Both embed Kleene-Kreisel functionals