On the Computational Content of the Axiom of Choice

Workshop in Honour of Thierry Coquand's 60th Birthday
Goteborg, Friday 26th August, 2022 11AM

S. Berardi

C. S. Dep., Torino University, Italy

C. S. Dept., Inner Yard, Dec. 12, 2019

The Plan of the Talk

This is **a survey talk** about almost a decade of work on constructivization of mathematics of S. Berardi, M. Bezem, D. Fridlender, under the guide of T. Coquand.

1. We first discuss the constructive interpretations of proofs using Excluded Middle and Choice, with a motivating example: Higman Lemma, a classical existence proof using choice axiom, whose constructive content was investigated in Fridlender's ph.d. thesis ([5]) supervised by T. Coquand.
2. Then we outline Coquand's game theoretical constructive interpretation of proofs ([4]).
3. Eventually, we sketch how this game interpretation was translated into a Realization interpretation of Excluded Middle and Choice by T. Coquand, S. Berardi and M. Bezem ([6]).

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§ 1. Constructive interpretations of proofs using Excluded Middle and Choice, with a motivating example: Higman Lemma.

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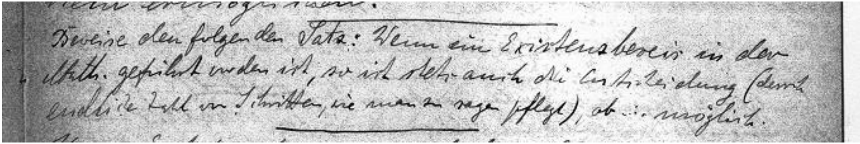
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Figure: **Hilbert Constructivization Conjecture** *(Courtesy from Goettingen State and University Library, Germany. Thanks to Susumu Hayashi for finding it, and to Benedikt Ahrens for translating)*.

The first known version (around 1917) of the following Constructivization Conjecture by Hilbert:

"Prove the following theorem: When a proof of existence has been concluded in mathematics, then also the decision (in a finite number of steps, as one says) is always possible. "

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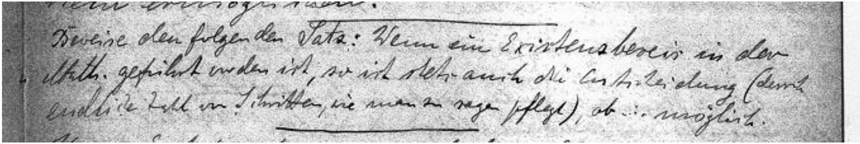

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Some Well-Known Facts about Hilbert Constructivization Conjecture

1. Turing proved that the original Hilbert conjecture is false.
2. Consider the following existence proof: for every computation of a Turing machine there is a boolean, which is "true" if the computation terminates and "false" if it runs forever.
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1. Hilbert conjecture's was restricted to the proofs of existence of some object with a *decidable* property.
2. In this form, it was proved true for plenty of formal systems for mathematics.
3. We will call any of such results a **constructive interpretation** for the formal system.
4. As example, we consider Spector Bar-Recursion [2] is an interpretation for Second Order Arithmetic with Excluded Middle and Choice, extending Kreisel Dialectica Interpretation for First Order.

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The constructive interpretation of the axiom of choice

The axiom of choice says that if $\forall x \in I.\exists y \in J.P(x,y)$, then there is a choice map f, taking any $x \in I$ and selecting some $y = f(x) \in J$ such that $P(x,y)$. The axiom of choice is a central tool in mathematical proof, but its constructive interpretation is difficult.

1. In constructive mathematics, the Axiom of Choice is validated by the Brouwer-Heyting-Kolmogorov explanation of the logical constants.
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Higman Lemma: A motivating Example for Interpreting Classical Choice

We state a miniature version of Higman's Lemma [1], an existence statment whose original proof used Classical Second Order Arithmetic and Choice Axiom. Assume that Σ is any finite alphabet and w, w' are words over Σ.

1. An embedding $f : w \to w'$ is an increasing map from $\{1, \ldots, \mathtt{l}(w)\}$ to $\{1, \ldots, \mathtt{l}(w')\}$, such that $w_i = w'_{f(i)}$ for all $i = 1, \ldots, \mathtt{l}(w)$. In this case we write $w \leq w'$.
2. An infinite sequence of words $\sigma = w_0, w_1, w_2, \ldots$ over Σ is *good* if for some $i < j$ we have $w_i \leq w_j$. Otherwise σ is *bad*.
3. For instance, if $\sigma_n = \langle\rangle$ for some $n \in N$ then $\sigma_n \leq \sigma_{n+1}$ and σ is good. If σ is bad then $\sigma_n \neq \langle\rangle$ for all $n \in N$.
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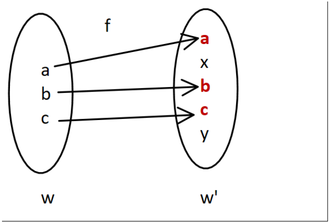
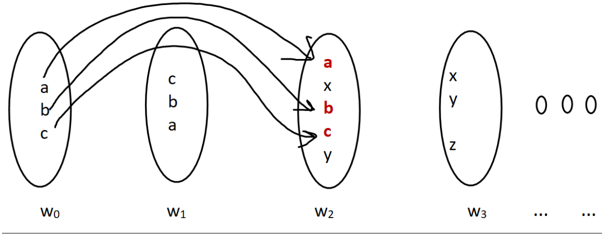
A motivating Example: Higman's Lemma



Embedding $f : w \rightarrow w'$



Good Sequence

There is no minimal bad sequence (Classical Proof)

We write $\sigma < \tau$ for: for some $n \in N$ we have $\mathtt{l}(\sigma_0) = \mathtt{l}(\tau_0), \ldots, \mathtt{l}(\sigma_{n-1}) = \mathtt{l}(\tau_{n-1})$ and $\mathtt{l}(\sigma_n) < \mathtt{l}(\tau_n)$.

1. Claim: there is no minimal for $<$ bad sequence of words on Σ.
2. Given $\sigma = \{\sigma_n\}_{n \in N}$ bad, we have $\sigma_n = a_n \tau_n$ for some sequence $\{a_n\}_n$ on Σ and some sequence of words τ.
3. Since Σ is finite, there is some $a \in \Sigma$ and some sub-sequence $a_{i_n} = a$ for all $n \in N$.
4. Let $\sigma | a = \sigma_0, \ldots, \sigma_{i_0 - 1}, \tau_{i_0}, \tau_{i_1}, \ldots$. Then $\sigma | a < \sigma$. By case analysis, if σ is bad then $\sigma | a$ is bad.
5. This is a non-constructive existence proof: there is no way of computing a out of σ, therefore no way of computing $\sigma | a$

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A proof of Higman Lemma using Classical Choice

Assume there is some bad sequence in order to derive a contradiction.

1. We already proved that there is no minimal bad sequence. If we are able to define a minimal bad sequence we get the desidered contradiction.
2. Given a bad sequence, we can define the minimal bad sequence using choice.
3. We choose any word w_0 of shortest length among those which are the first word of a bad sequence.
4. We choose any word w_1 of shortest length among those which are the second word of a bad sequence whose first word is w_0.
5. We define in this way a sequence $\sigma = w_0, w_1, w_2, \ldots$.
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5. We define in this way a sequence $\sigma = w_0, w_1, w_2, \ldots$.
6. We easily check that σ is bad and minimal among bad sequences, contradiction.

A proof of Higman Lemma using Classical Choice

Assume there is some bad sequence in order to derive a contradiction.

1. We already proved that there is no minimal bad sequence. If we are able to define a minimal bad sequence we get the desidered contradiction.
2. Given a bad sequence, we can define the minimal bad sequence using choice.
3. We choose any word w_0 of shortest length among those which are the first word of a bad sequence.
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Is Choice required to prove Higman Lemma ?

As it is often the case in mathematical proofs, Choice is not really required but it is useful to have.

1. Whenever we have to choose some word w with a given property P, we can choose the smallest w in the lexicographic order such that $P(w)$.
2. In this was the choice map can be defined and proved total using Excluded Middle only.
3. However, the extra criterion making the choice unique has nothing to do with the proof.
4. In the constructive interpretation, the extra criterion requires a large overhead of work. It is not enough to provide some w such that $P(w)$, we have to try several w such that $P(w)$ in order to find the the smallest such w in the lexicographic order.

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The construction in Higman Lemma

In the particular case of Higman Lemma, the following construction was found in [5].

1. Assume σ is any infinite sequence of words on a finite alphabet Σ. For $x \in \Sigma$, let σ_x be defined as in the slide "There is no minimal bad sequence".
2. We compute (in interleaving) all decreasing chains $\sigma > \sigma|a > (\sigma|a)|b > ((\sigma|a)|b)|c > \ldots$ for any $a, b, c, \ldots \in \Sigma$, trunking at the same finite prefix of σ.
3. We stop when we find some subsequence $((\sigma|a)|b)|\ldots$ with an empty word followed by some word.
4. We have an embedding in $((\sigma|a)|b)|\ldots$ and we define an embedding in σ from it.
5. We prove termination for this algorithm either directly, or from the general properties of the interpretation we are using.

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§ 2. A costructive game interpretation of Excluded Middle and Choice

Coquand game theoretical interpretation of classical choice, by trial-and-error

In [4], Coquand interprets the truth of any disjuntion on a list $\Gamma = A_1, \ldots, A_n$ of *closed* second order arithmetical formulas through a game between Eloise, asserting the truth of some $A_i \in \Gamma$, and Abelard, asserting the falsity of all $A_i \in \Gamma$.

1. Eloisa chooses either some disjunctive $A_i = A_{i,1} \vee A_{i,2}, \exists x.B$, then some instance $A_{i,j}, B[j/x]$ and asserts it to be true, or
2. Eloise chooses some conjunctive $A_i = A_{i,1} \wedge A_{i,2}, \forall x.B$, and in this case Abelard choose some instance $A_{i,j}, B[j/x]$ and asserts it to be false.

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What is new in Coquand's game theoretical interpretation

The difference with the usual interpretation is that Eloise (not Abelard) can suspend the attempt to assert A_i and can switch to another A_j, using the experience gathered in defending A_i in order to better defend A_j. This operation is called *backtracking*.

1. Eloise can resume any suspended attempt from the sub-formula in which she suspended it.
2. Eloisa wins if eventually she asserts the truth of a true closed atomic formula, otherwise Abelard wins.
3. Any proof with Excluded Middle can be interpreted by a winning strategy for Eloise.
4. This is a constructive interpretation of Excluded Middle, that is, an effective interpretation for proofs of existences of objects with a decidable property.

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Eloise has a winning strategy for the Axiom of Choice
$\forall x.\exists y.P(x,y) \rightarrow \exists f.\forall x.P(x,f(x))$.

1. By classical logic, the Axiom of Choice is written $\Gamma, \exists x.\forall y.\neg P(x,y), \exists f.\forall x.P(x,f(x))$.
2. Eloise's goal is finding some x_i, f_i such that Abelard asserts both $\neg P(x_i, f_i(x_i))$ and $P(x_i, f_i(x_i))$. This is an instance of Excluded Middle: eventually, Eloise will apply a winning strategy for Classical Login and she wins.
3. Eloise first chooses $\exists f.\forall x.P(x,f(x))$, then $f = f_0$, any dummy map. Abelard chooses some x_0 and asserts that $P(x_0, f_0(x_0))$ is false.
4. Eloise changes her choice to $\exists x.\forall y.\neg P(x,y)$, then $\forall y.\neg P(x_0, y)$, and Abelards asserts that $\neg P(x_0, y_0)$ is false.

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1. Eloise redefines f_0 to $f_1(x) =$ if $x = x_0$ then y_0 else $f_0(x)$ and restarts the cycle, choosing $\exists f.\forall x.P(x, f(x))$, then $f = f_1$.
2. Abelard chooses some x_1 and asserts that $P(x_1, f_1(x_1))$ is false. Eloise changes her choice to $\exists x.\forall y.\neg P(x, y)$, then $\forall y.\neg P(x_1, y)$, and Abelards asserts that $\neg P(x_1, y_1)$ is false.
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4. By a continuity argument we have $x_{i+1} = x_i$ for some i, therefore Abelards asserts both $\neg P(x_i, y_i)$ and $P(x_{i+1}, f_{i+1}(x_{i+1})) = P(x_i, f_{i+1}(x_i)) = P(x_i, y_i)$. Now Eloise is able to win using a winning strategy for Excluded Middle.
5. This game interpretation can be translated into the programming language $\mathcal{P}$.

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§3. A game interpretation of Excluded Middle and Choice translated into a Realization interpretation for the same principles

The paper "On the computational content of the axiom of choice"

This is a 1996 paper by by S.Berardi, M.Bezem and T.Coquand [6].

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2. Coquand's interpretation is computationally more direct than Godel's Dialectica interpretation, and the resulting algorithm, based on trial-and-error game interpretation of classical logic, is more intuitive than Bar Recursion.

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Coquand's interpretation of Choice [6]

We start defining a programming language $\mathcal{P}$ for interpreting the constructions of higher-order-constructive arithmetic HA^{ω}.

1. Types are $\mathtt{N}, \mathtt{Unit}, \mathtt{Abs}$ and with τ, τ' also $\tau \to \tau'$, $\tau \times \tau'$ (cartesian product) and $[\tau]$ (lists over type τ).
2. constants R^{τ} for primitive recursion of type τ, $()$: $Unit$, $Dummy : Abs$, $Axiom_1, Axiom_2 : N \to Abs$, constants for general recursion (fixpoint combinators of all appropriate types) and constants for pairing and projection and list construction and destruction.
3. The term ($\mathtt{get}$ x l a) searches the list l for the first triple whose first component matches x; if such a triple is found, then f is applied to the second and third component of the triple, otherwise the output is a.

The Realization Interpretation for costructive proofs

There is a mapping ϕ from formulas ϕ of HA^ω to types $|\phi|$ of $\mathcal{P}$. Any proof $p : \phi$ of HA^ω is turned into a term $|p| : |\phi|$ of $\mathcal{P}$, representing its constructive content.

1. $|M = M'| = Unit$
2. $|\bot| = Abs$
3. $|\phi \to \psi| = |\phi| \to |\psi|$
4. $|\phi \wedge \psi| = |\phi| \wedge |\psi|$
5. $\forall x : \tau\phi = \tau \to |\phi|$
6. $\exists x : \tau\phi = \tau \wedge |\phi|$

The Realization Interpretation for proofs with Excluded Middle

1. We use negative interpretation for classical logic. We replace each $\vee, \exists$ in each formula in the proof with $\neg\neg\vee$, $\neg\neg\exists$. If we start from an existence proof of an object with a decidable property, say a proof of $\exists x f(x) = 0$, we obtain a proof p of $\neg\neg\exists x f(x) = 0$, then of

$$\neg\forall x(f(x) = 0 \to \bot)$$

2. We define a realizer of $\forall x(f(x) = 0 \to \bot)$ from $\mathtt{axiom}_1$:

$$r = \lambda x, h.\ \mathtt{if}\ f(x) = 0\ \mathtt{then}\ \mathtt{axiom}_1(x)\ \mathtt{else}\ \mathtt{dummy}$$

We prove that $p(r) : \bot$ reduces to some $\mathtt{axiom}_1(n)$ such that $f(n) = 0$. That is, we provided a construction returning some n such that $f(n) = 0$.

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The Realization Interpretation for proofs with Excluded Middle and Choice

We have to define a realizer r for the negative interpretation of choice: $\forall x.\neg\neg\exists y.\neg\phi(x, y) \rightarrow \neg\neg\exists f.\forall x.\neg\phi(x, f(x))$. r translates Eloise's winning strategy for classical logic into lambda calculus.

1. r takes a finite list l of triples $\langle x_i, y_i, q_i \rangle$ with q realizer of $\neg\phi(x, y)$, a realizer p of $\neg\neg\exists y.\neg\phi(x, y)$, a realizer h of $\neg\exists f.\forall x.\neg\phi(x, f(x))$.
2. From l we define a map $f = fun(l)$ sending any x_i to y_i and any other x to *dummy* and a partial realizer s of $\forall x.\neg\phi(x, f(x))$, valid for $x = x_i$ for some i.
3. r applies h to f and a partial realizer of $\forall x.\neg\phi(x, f(x))$.

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The Realization Interpretation for proofs with Excluded Middle and Choice

1. If h never requires an instance of s on some $x \neq x_i$ for all i then we have a realizer.
2. Otherwise, r asks p to provide for x a realizer q of $\neg\phi(x, y)$.
3. Then the process restarts with the list l extended with the triple $\langle x, y, q\rangle$.
4. By a continuity argument eventually the list l stops growing and indeed we have a realizer of choice.

End Of the Talk

I want to thank the organizer of the Workshop in Honour of Thierry Coquand's 60th Birthday, for giving me the possibility of reliving the joint works I had with T. Coquand and with more friends, M. Bezem and D. Fridlender.

I hope I could communicate to the audience the interest of a trial-and-error constructive interpretation, valid for the most of Classical Mathematics, and first proposed by T. Coquand.

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