

Thierry's Contributions to HoTT

- 2006: Workshop in Uppsala (Palmingren)
- 2007: "HoTT" in Nancy...
- 2009: Martin-Löf Colloquium in Uppsala (Aczel, Tait, ...)
TC: "Forcing in type theory", SA: HoTT
- 2010: Chiemsee Workshop on Constr. TT. (Schwichtenberg)
VV, TC, SA, ...
- 2011: Oberwolfach: VV states homot. canon. conj., HITS,
CoqHoTT, IAS special year to be org. by VV, TC, SA.
- 2012/13: IAS Year: TC worked hard on constr. model of UA.
MB & TC: sSet not constructive.
- 2014: BCH constr. model of UA in cSet.
- 2016: CCHM: Cubical TT, SH: Canonicity, Proof of VV's conj.
- 2016 ≤: Cubical Homotopy, QMS, stack models, ...
+ Cubical Agda...

Algebraic Type Theory

Steve Awodey

in honor of Thierry Coquand's
60th Birthday

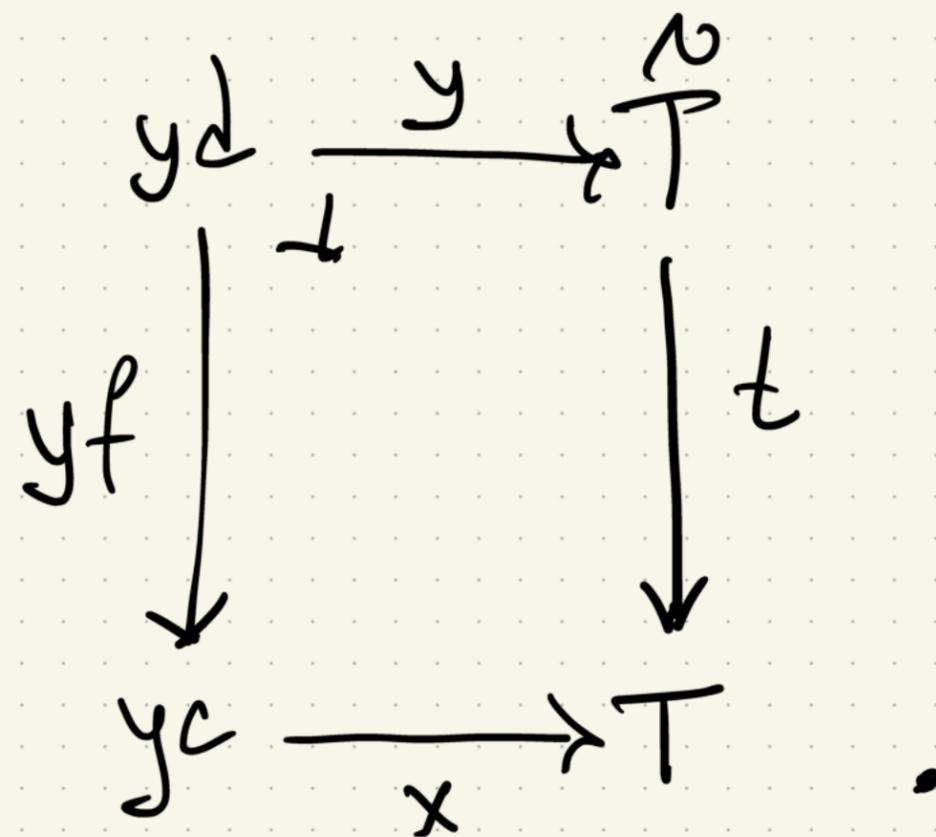
August 2022
Gothenburg

1. Natural Models of DTT

Def. (A.2012) A natural model consists of:

- a Cat \mathcal{C}
- presheaves T, \tilde{T}
- a nat. transf. $t: \tilde{T} \rightarrow T$
that is representable:

$$\forall c \in \mathcal{C} \quad \forall x \in T_c \quad :$$
$$\exists f: d \rightarrow c \quad \exists y \in \tilde{T}_d$$



Remarks

- (1) • \mathcal{C} cat of ctx's
• \mathcal{T} presheaf of types
• $\hat{\mathcal{T}}$ presheaf of terms

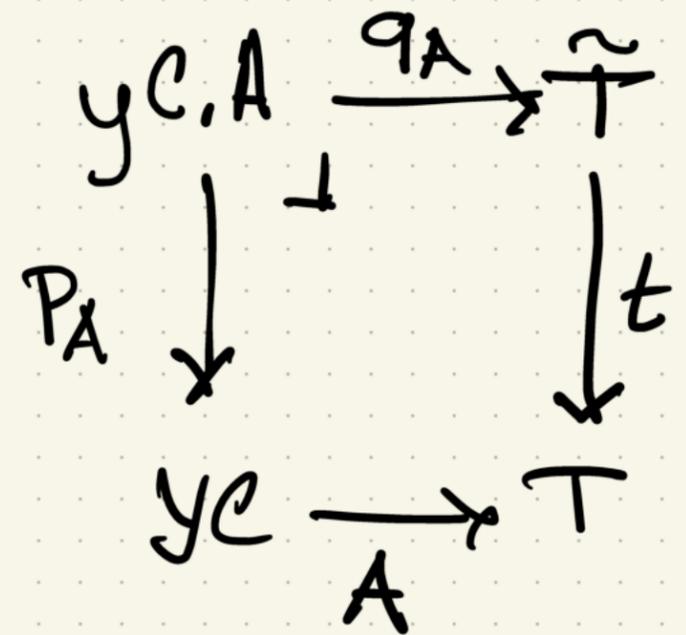
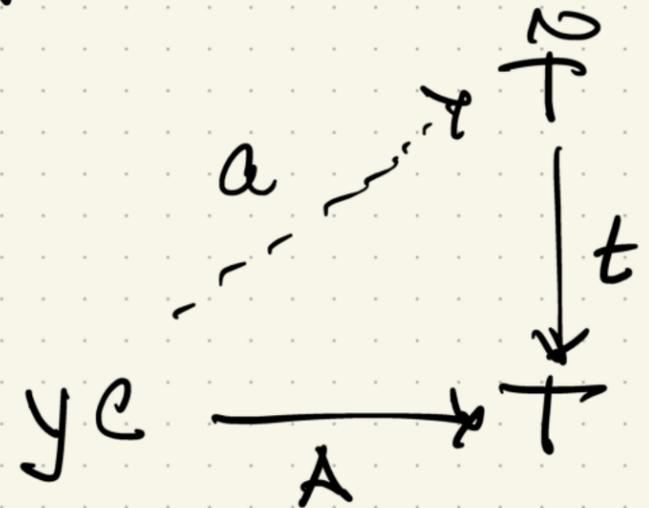
(2) Representability is ctx. extension:

(3) This is equivalent to CWF.

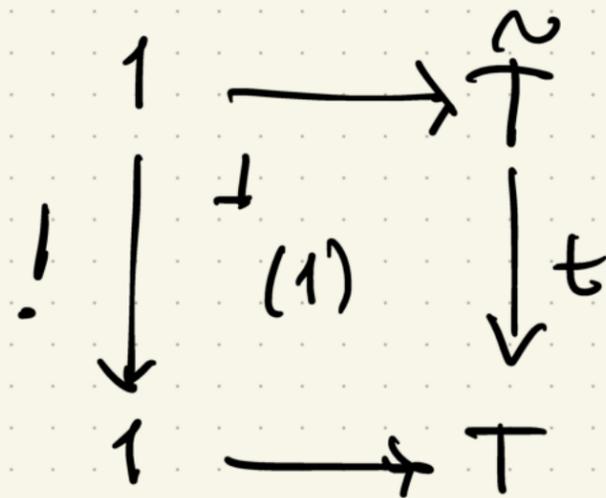
(4) (A.2012) gives conditions on t
equivalent to the CWF having
 $1, \Sigma, \Pi, Eq, Id$

$\mathcal{C} \vdash a:A$

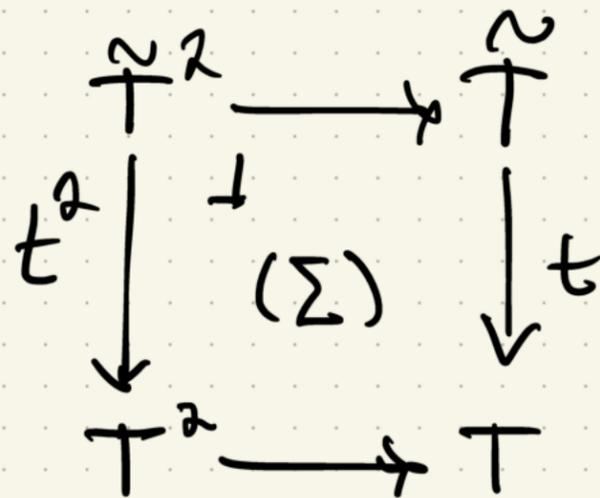
\Leftrightarrow



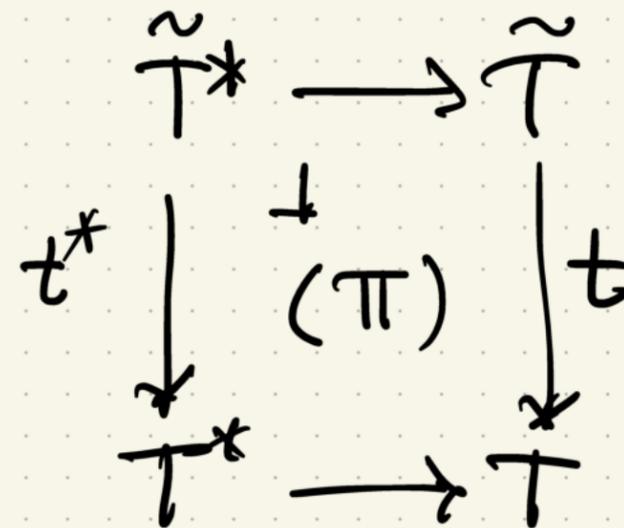
(5) Namely, e.g.:



unit type



dependent sum



dependent product

(6) We shall abstract this structure to form that of a "Martin-Löf algebra".

2. Polynomial Functors

Let \mathcal{E} be LCCC.

Every $f: A \rightarrow B$ determines a polynomial functor

$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{P_f} & \mathcal{E} \\ A^* \downarrow & & \uparrow B! \\ \mathcal{E}/A & \xrightarrow{f_*} & \mathcal{E}/B \end{array}$$

$$\begin{array}{ccc} X & \longleftarrow & X \times A \\ & & \downarrow \\ & & A \xrightarrow{f} B \\ & & \uparrow P_f X \\ & & \downarrow \end{array}$$

2. Polynomial Functors

(1) In the internal DTT of \mathcal{C} :

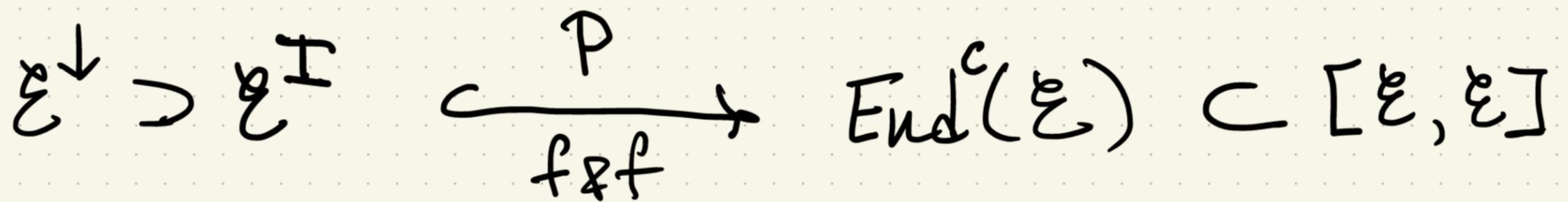
$$P_f(X) = \mathcal{B}! f_* A^\otimes(X) = \mathcal{B}! f_* f^* A^\otimes(X) = \sum_{b: \mathcal{B}} X^{A_b}.$$

(2) UMP of $P_f X$ is:

$$(b, x): Z \longrightarrow P_f X$$

$$\begin{array}{ccccc} X & \xleftarrow{x} & A_b & \xrightarrow{\quad} & A \\ & & \downarrow & & \downarrow f \\ & & Z & \xrightarrow{b} & \mathcal{B} \end{array}$$

(3) The assignment $f \mapsto P_f$ is functorial on pullbacks:



(4) The composite of polynomial functors is polynomial:

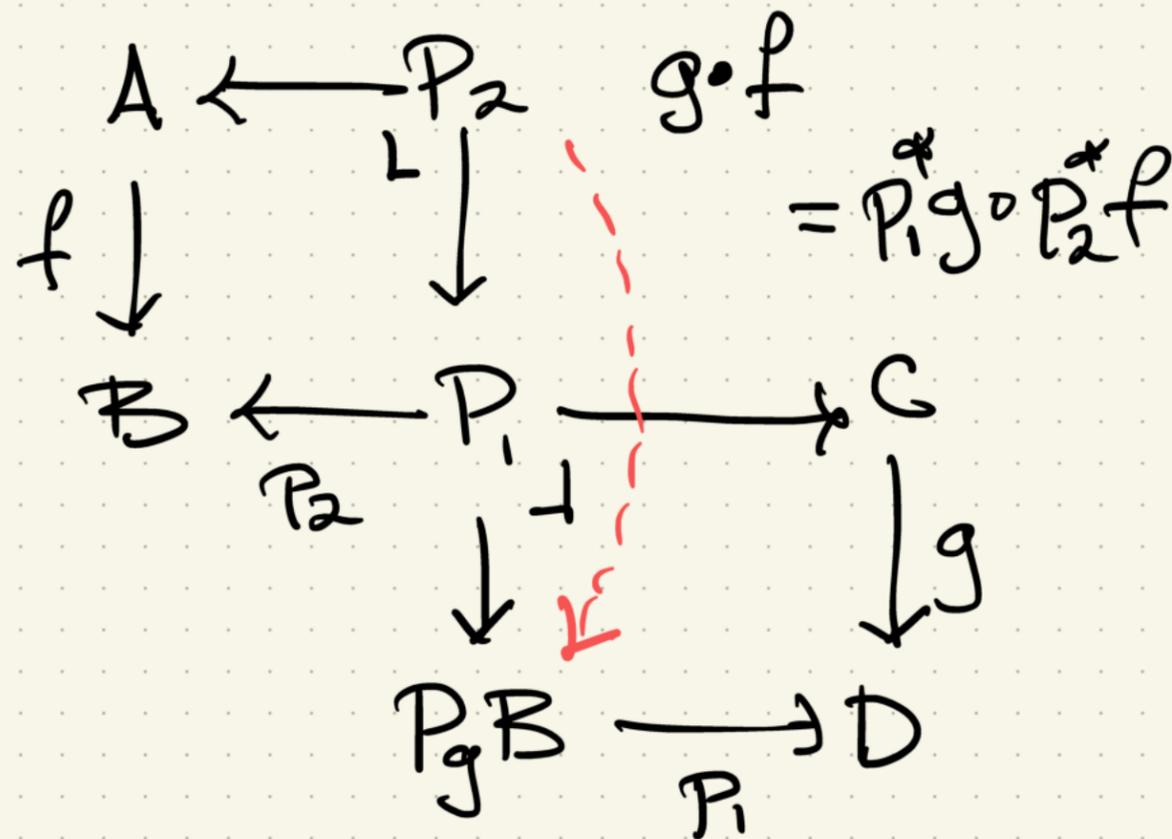
$$\begin{array}{ccc}
 \mathcal{E} & \xrightarrow{P_{g \circ f}} & \mathcal{E} \\
 P_f \searrow & & \nearrow P_g \\
 & \mathcal{E} &
 \end{array}$$

$$\begin{array}{ccc}
 A & C & E \\
 f \downarrow & g \downarrow & \downarrow g \circ f \\
 B & D & F
 \end{array}
 \rightsquigarrow$$

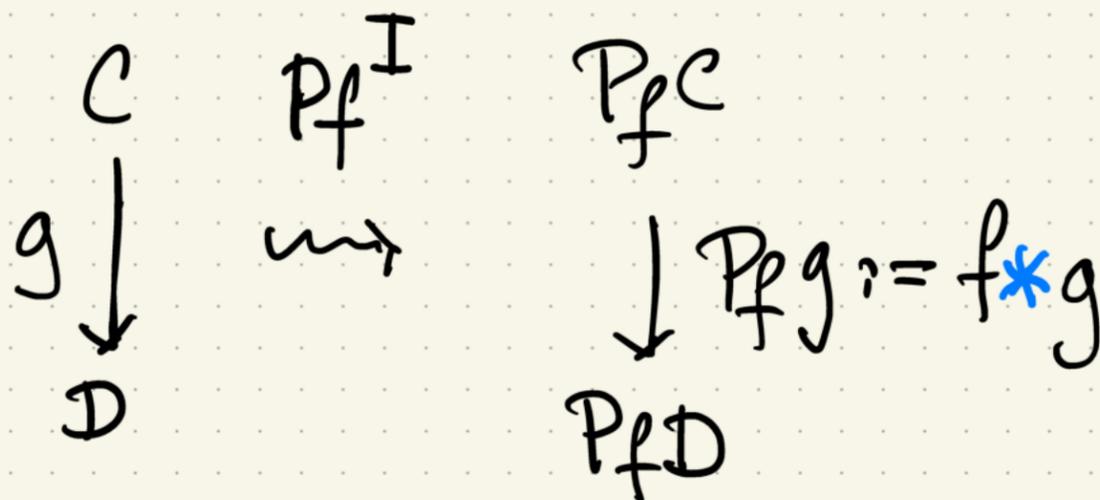
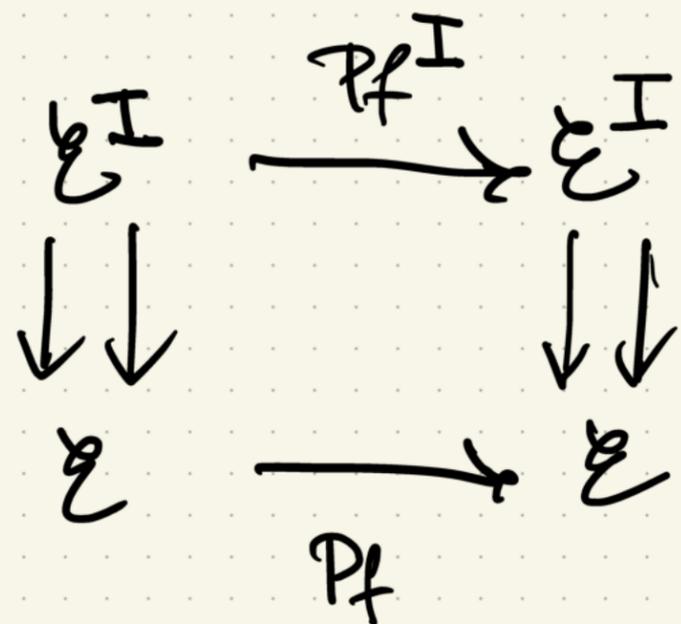
And $P\left(\begin{smallmatrix} \uparrow \\ ! \downarrow \\ \uparrow \end{smallmatrix}\right) = I_{\mathcal{E}}$, so there is an equivalence of monoids:

$$(\mathcal{E}^I, \cdot, !) \cong (\text{Poly}(\mathcal{E}), \circ, I_{\mathcal{E}})$$

(5) Construction of $g \circ f$:



(6) Polynomials preserve pullbacks, so they lift to \mathcal{E}^I :

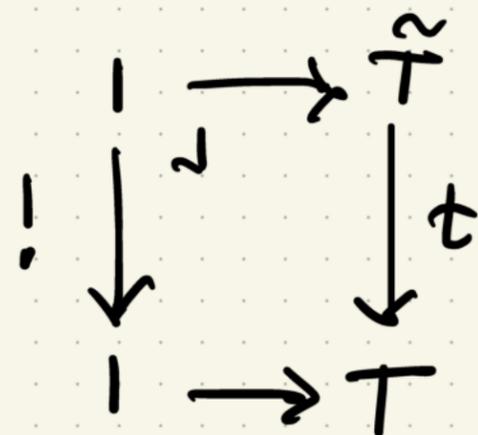


3. M-L Algebras

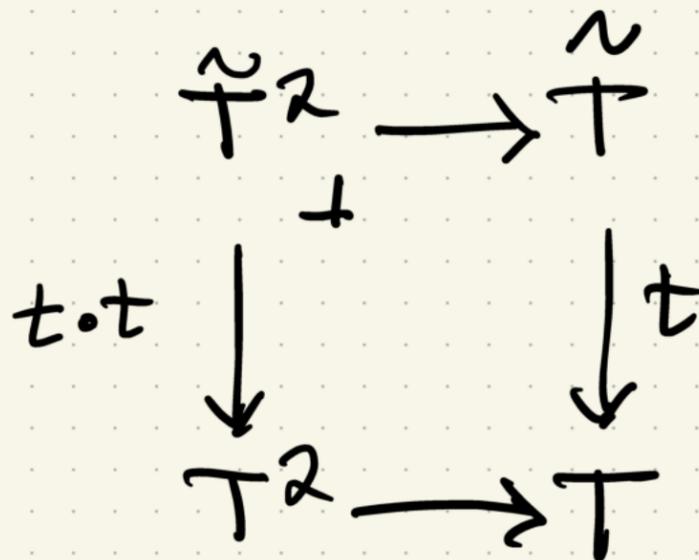
Def. A M-L algebra in a LCC \mathcal{E} is a map

$$t: \tilde{T} \rightarrow T$$

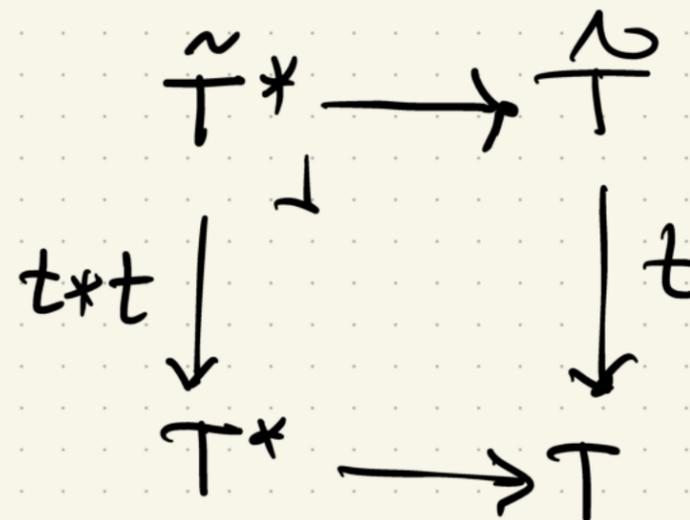
with structure:



unit



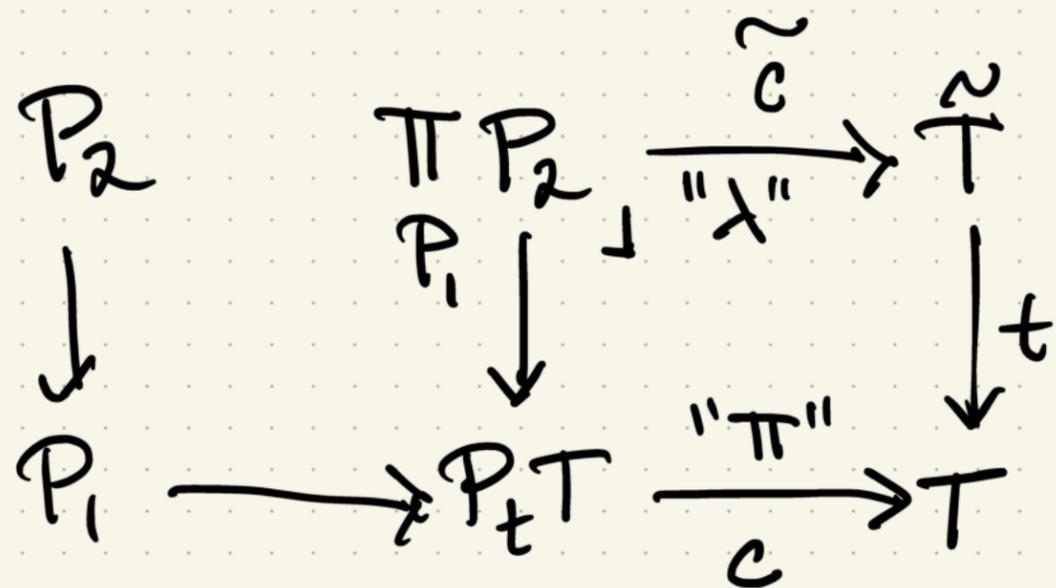
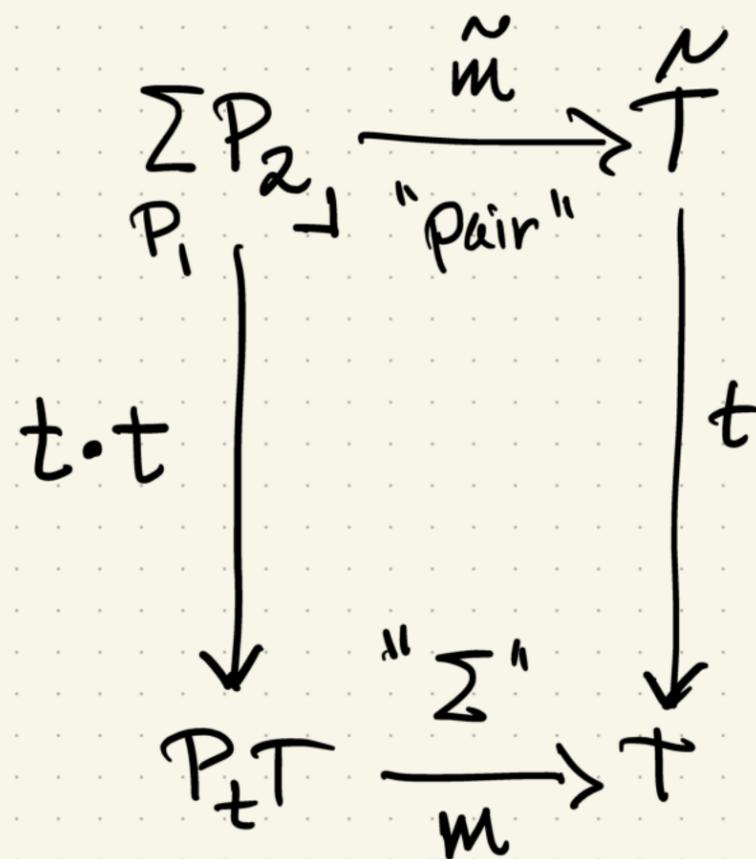
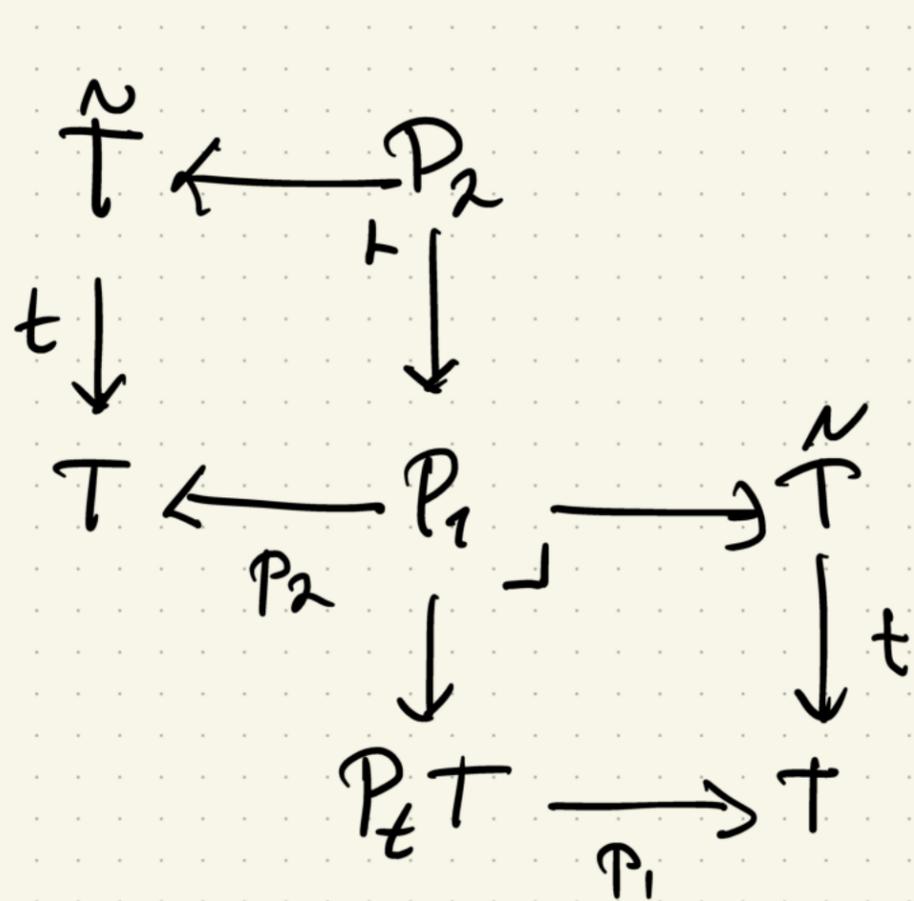
multiplication



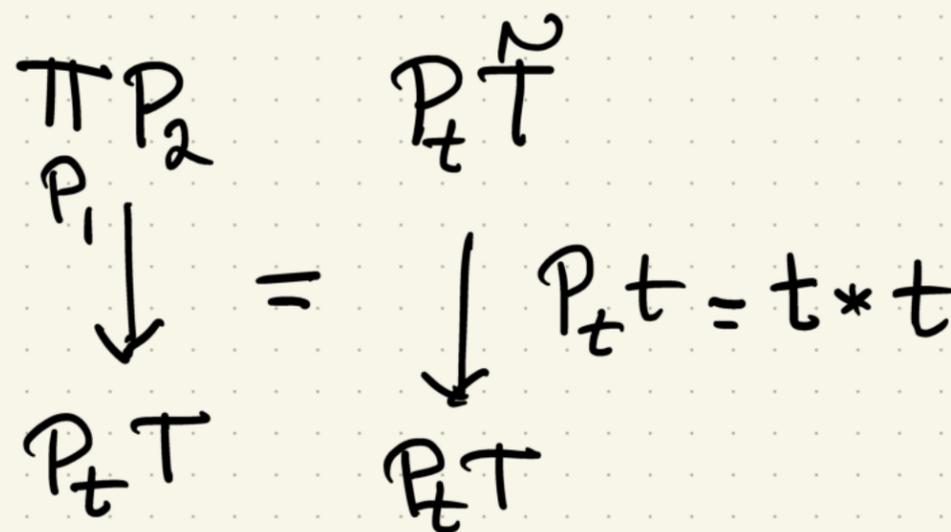
closure

dominance

The general pattern:



Note:



- The unit determines a cart. nat. trans.

$$u: 1_{\xi} \rightarrow P_t$$

- The mult. determines

$$m: P_t \circ P_t \rightarrow P_t$$

- The closure determines an algebra structure

$$c: P_t(t) \rightarrow t$$

- In terms of the monoidal cat $(\xi^I, \cdot, !)$:

- a monoid structure $! \rightarrow t \leftarrow t \cdot t$

- a module structure $t \times t \rightarrow t$

Basic Example: A CwF $(\mathcal{C}, t: \tilde{T} \rightarrow T)$ is a ML-algebra in $\hat{\mathcal{C}}$ iff it has $1, \Sigma, \Pi$.

Conversely:

Thm Let $t: \tilde{T} \rightarrow T$ be a ML-algebra in \mathcal{E} .
Define a CwF on \mathcal{E} by "mapping in":

$$\begin{array}{ccc} \tilde{T}' & = & \text{Hom}_{\mathcal{E}}(-, \tilde{T}) \\ t' \downarrow & & \downarrow \text{Hom}_{\mathcal{E}}(-, t) \\ T' & = & \text{Hom}_{\mathcal{E}}(-, T) \end{array}$$

Then $t' = yt$ has $1, \Sigma, \Pi$ as a CwF.

(pf: Yoneda preserves ML alg.s.)

4. Examples

(i) Display maps. Take any map $t: \hat{T} \rightarrow T$ in \mathcal{E} and define display maps $\mathcal{D}_t \subseteq \mathcal{C}_1$ by:

$$\begin{array}{ccc} \mathcal{D} & & \mathcal{D} \\ d \downarrow \in \mathcal{D}_t & \Leftrightarrow & \begin{array}{ccc} \mathcal{D} & \xrightarrow{\quad} & \hat{T} \\ d \downarrow & \exists & \downarrow t \\ E & \xrightarrow{\quad} & T \end{array} \\ E & & \end{array}$$

Then \mathcal{D}_t is closed under pullbacks, and

- under isos & composition if t is a dominance,
- under pushforwards if t is closed.

So $(\mathcal{C}, \mathcal{D}_t)$ is a π -clan* (Joyal) if t is a ML algebra,

Conversely:

Thm (A.2012) Given a display map $\text{cat}(\mathcal{C}, \mathcal{D})$,
there's a $d_{\mathcal{D}}: \tilde{\mathcal{D}} \rightarrow \mathcal{D}$ in $\hat{\mathcal{C}}$ that's a ML-algebra
if $(\mathcal{C}, \mathcal{D})$ is closed under isos, composition,
and pushforwards, i.e. if $(\mathcal{C}, \mathcal{D})$ is a Π -clan.

In fact:

$$\begin{array}{ccc} \tilde{\mathcal{D}} & & \coprod_{d \in \mathcal{D}} \mathcal{D} \\ \downarrow d_{\mathcal{D}} & \stackrel{:=}{=} & \downarrow y_d \\ \mathcal{D} & & \coprod_{d \in \mathcal{D}} \mathcal{D} \end{array}$$

So $\mathcal{D}_{d_{\mathcal{D}}} = \mathcal{D}$.

(ii) Finite sets, In $\mathcal{E} = \text{Set}$, let

$$\begin{array}{ccccc} \tilde{\mathbb{N}} & & \sum_{n:\mathbb{N}} [n] & & \mathbb{N} \\ \text{nat} \downarrow & = & \downarrow & = & \downarrow \\ \mathbb{N} & & \mathbb{N} & & \mathbb{N} \end{array}$$

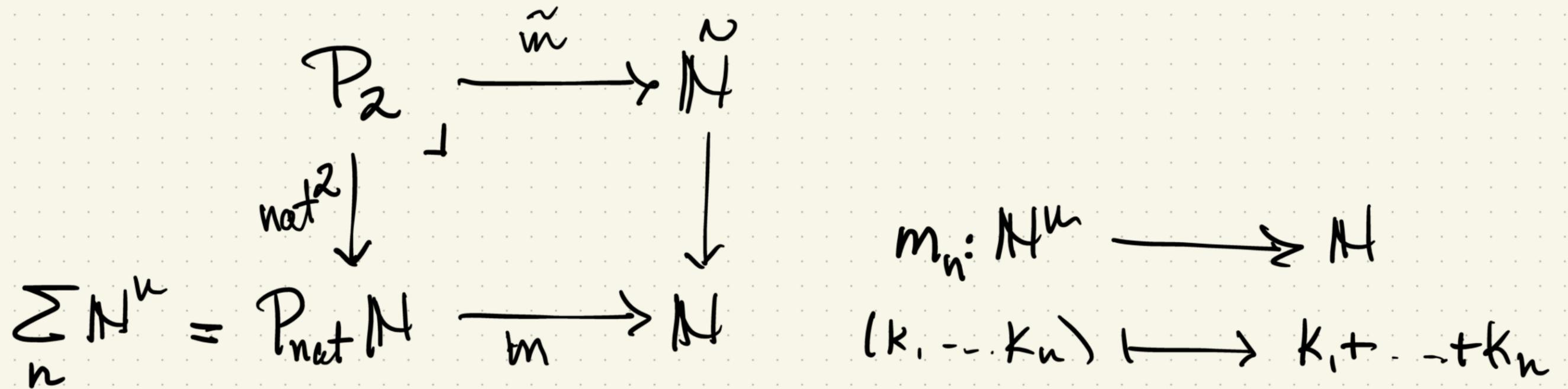
$\mathbb{N} \times \mathbb{N}$
 $\downarrow P_2$

Polynomial functor $P_{\text{nat}}: \text{Set} \rightarrow \text{Set}$ is then

$$\begin{aligned} P_{\text{nat}}(X) &= \sum_{n:\mathbb{N}} X^n \\ &= 1 + X + X^2 + \dots \end{aligned}$$

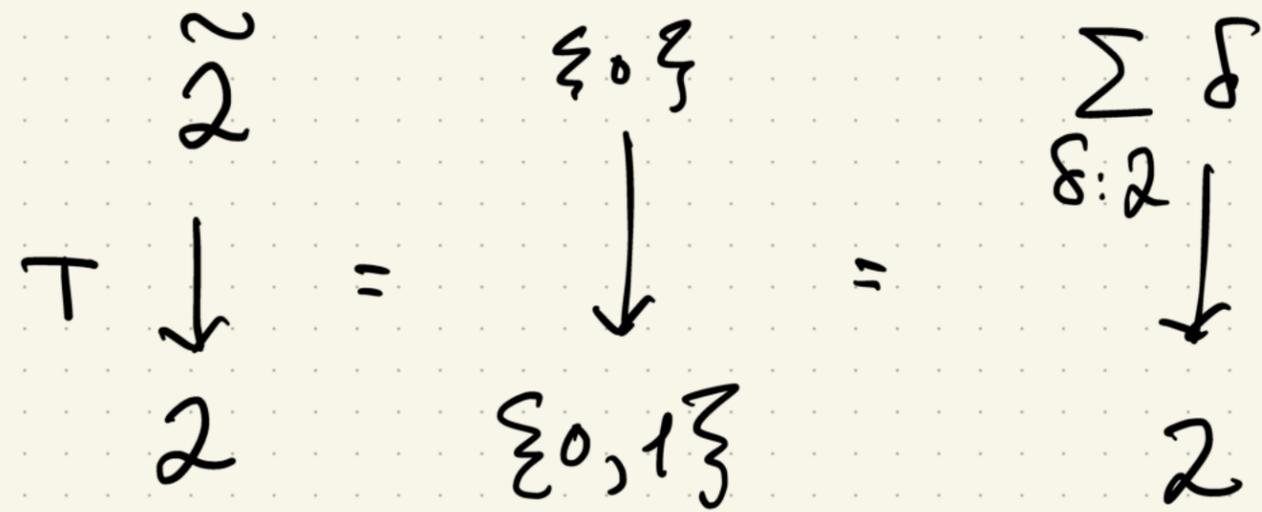
• unit : $u_X : X \rightarrow 1 + X + \dots$ (+ inclusion)

• multiplication : $P_{nat}^2 \rightarrow P_{nat}$
 $P_{nat} \circ_{nat} X$



• Closure : $C_n(k_1, \dots, k_n) = k_1 \cdot \dots \cdot k_n$

(iii) Bool.



$$P_{\mathbb{T}}(X) = \sum_{\delta: \mathbb{2}} X^{\delta} = 1 + X.$$

• unit: $X \longrightarrow 1 + X$ +-incl.

• mult: $1 + (1 + X) \longrightarrow 1 + X$ ∇

• closure:
$$\begin{array}{ccc} 1 + 1 & \longrightarrow & 1 \\ \downarrow & + & \downarrow \\ 1 + 2 & \longrightarrow & 2 \end{array}$$
 Soc

(iv) Groth Universe.

Take any cardinal $\alpha \neq \aleph_0$ do "the same thing":

\tilde{S}_α " $\sum_{a \in S_\alpha} a$ "

↓

S_α "sets of size $\leq \alpha$ "

ML-algebra

if α is inaccessible.

(v) Syntactic ML-algebra of DTT w/ λ, Σ, Π :

$\hat{\Gamma}$
(ctx)

Terms

$\{c \vdash a : A\}$

↓
Types

$\{c \vdash A \text{ type}\}$

Should be

the initial

ML-algebra!

(vi) Hofmann-Streicher Universe,

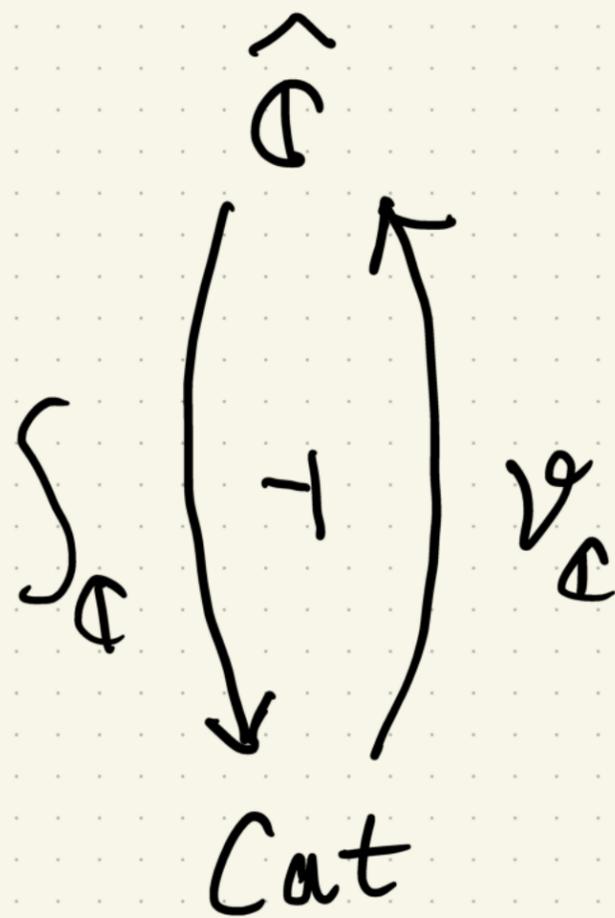
Given a cardinal $\alpha \in \text{Set}$, for any \mathcal{C} ,
we have the H-S universe in $\hat{\mathcal{C}}$:

$$\begin{array}{ccc} \mathcal{U} : \mathcal{C}^{\text{op}} \longrightarrow \text{Set}_\alpha & & \tilde{\mathcal{U}}_\alpha \\ \mathcal{E} : \sum_{\mathcal{C}} \mathcal{U}^{\text{op}} \longrightarrow \text{Set}_\alpha & \cong & \downarrow \\ & & \mathcal{U}_\alpha \end{array}$$

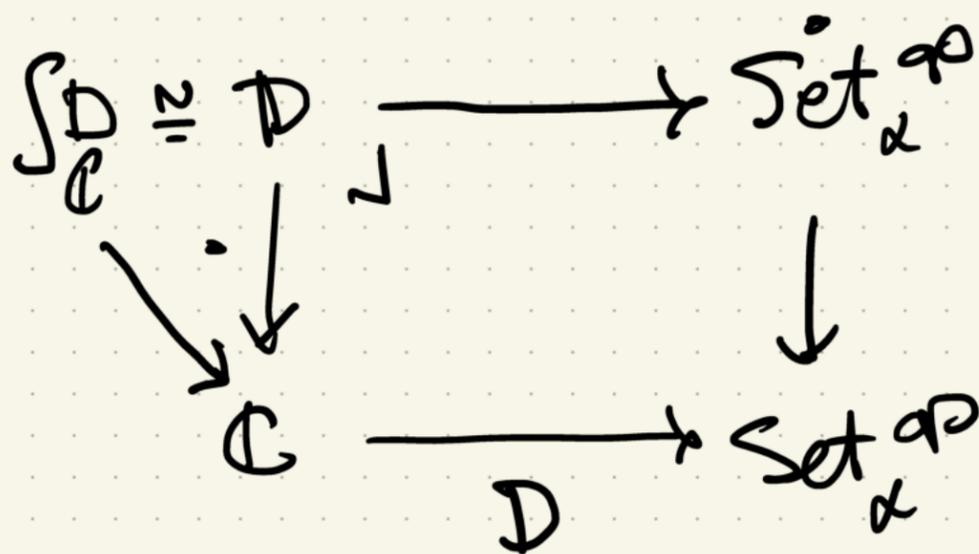
Prop. $\tilde{\mathcal{U}}_\alpha \rightarrow \mathcal{U}_\alpha$ is a ML-algebra (for suitable α).

This follows directly from the 3 facts...

Fact 1: $\tilde{U}_\alpha \rightarrow U_\alpha$ is the "nerve" of the universal α -small discrete fibration $\mathring{\text{Set}}_\alpha \rightarrow \text{Set}_\alpha$ in Cat :



$$\begin{array}{ccc} \tilde{U}_\alpha & & \mathring{\text{Set}}_\alpha \\ \downarrow & = & \downarrow \\ U_\alpha & & \text{Set}_\alpha \end{array}$$



Fact 2: The nerve $\mathcal{V}_{\mathcal{C}}: \text{Cat} \longrightarrow \hat{\mathcal{C}}$ preserves
ML-algebras.

Fact 3: $\text{Set}_{\alpha}^{\text{op}} \longrightarrow \text{Set}_{\alpha}^{\text{op}}$ is a ML-algebra in Cat^*
(for suitable α).

So indeed:

Prop. $\tilde{\mathcal{U}}_{\alpha} \longrightarrow \mathcal{U}_{\alpha}$ is a ML-algebra in $\hat{\mathcal{C}}$
(for suitable α).

Fun Corollary: the Soc $1 \rightarrow \Omega$ in $\hat{\mathcal{P}}$
is also an MC-algebra.

Because it's the nerve of $T: 1 \rightarrow 2$,

$$\begin{array}{ccc} 1 & & 1 \\ \downarrow t & = & \downarrow v^T \\ \Omega & & 2 \end{array}$$

5. Free Completions

Let $(\mathcal{C}, t: \tilde{T} \rightarrow T)$ be a CwF in $\mathcal{E} = \hat{\mathcal{C}}$.

- We saw that TFAE:

ML-Alg	CwF
$! \rightarrow t$	unit type $\ast : 1$
$t \cdot t \rightarrow t$	sum type $\Sigma_A B$
$t \ast t \rightarrow t$	product type $\Pi_A B$

- Given any CwF t , we can freely add these structures to make the free ML-algebra on t .

Step 1 Using $P_f + P_g = P_{f+g}$ in $\text{Poly}(\mathcal{E}) \simeq \mathcal{E}^I$, we have

$$\text{in Poly}(\mathcal{E}) \quad 1_{\mathcal{E}} \longrightarrow 1_{\mathcal{E}} + P_t \longleftarrow P_t \quad ,$$

in \mathcal{E}^I

$$\begin{array}{ccccc} 1 & \longrightarrow & 1 + \hat{T} & \longleftarrow & \hat{T} \\ \downarrow & & \downarrow !+t & & \downarrow t \\ 1 & \longrightarrow & 1 + T & \longleftarrow & T \end{array} \quad .$$

Since $1_{\mathcal{E}} + P_t = P_{!+t}$ is the free pointed endofunctor on P_t ,
the map $t \rightarrow !+t$ freely adds a unit type to the CwF t .

Step I: We seek u in \mathcal{E}^I with

$$t \xrightarrow{\cdot} u \xleftarrow{\cdot} u \circ u \quad (\text{universal})$$

- As in the theory of endofunctors as "data types", this is given by solving the "domain equation"

$$u \cong 1 + u \circ t$$

This also adds a point $1 \rightarrow u$.

- The solution is the lim of the sequence

$$0 \rightarrow T_0 \rightarrow T^2_0 \rightarrow \dots,$$

for the endofunctor

$$T(x) = 1 + x \circ t$$

on $\mathcal{E}^I \cong \text{Poly}(\mathcal{E})$.

The colimit in \mathcal{E}^I is seen to be the object

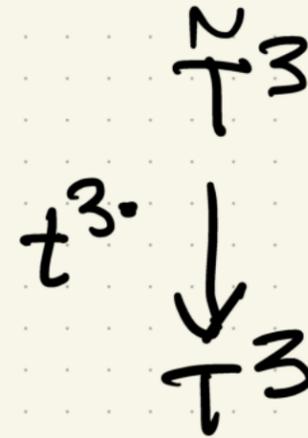
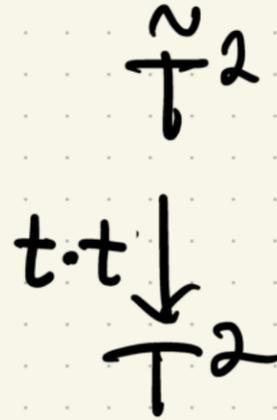
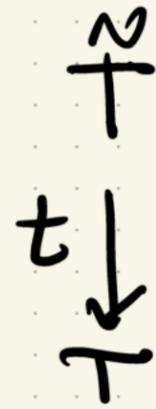
$$\begin{aligned} u &= ! + t + t \circ t + t^{\circ 3} + \dots \\ &= \sum_n t^{\circ n} \end{aligned}$$

For this, one uses the following important

Fact: If $t: \hat{T} \rightarrow T$ in $\hat{\mathcal{C}}$ is representable, then the polynomial functor $P_t: \hat{\mathcal{C}} \rightarrow \hat{\mathcal{C}}$ has a right adjoint, and therefore preserves all colimits. So in particular:

$$t \circ (a + b) \cong t \circ a + t \circ b, \quad a, b \in \mathcal{E}^I.$$

Objects:



...

classify:

A



...

ctx's:

A

A.B

A.B.C

A.B.C.D

...

- Thus $u: \tilde{\mathcal{U}} \rightarrow \mathcal{U}$ classifies the ctx's of t :

$$\underbrace{\quad}_{(1+t+t^2+\dots)}$$

$$\begin{array}{ccc}
 C_n & \xrightarrow{\quad} & \tilde{\mathcal{U}} \\
 \downarrow & \lrcorner & \downarrow u \\
 C_0, C_1, \dots, C_n & \vdots & \\
 \downarrow & & \\
 C_0 & \xrightarrow{\quad} & \mathcal{U}
 \end{array}$$

- This agrees with the idea that the cat. of ctx's of a theory "freely adds Σ -types" (cf. GFG).

- Can be used to give a "base change" of CwFs:

$$(\mathbb{C}, t) \longrightarrow (\mathbb{C}^\Sigma, t^\Sigma)$$

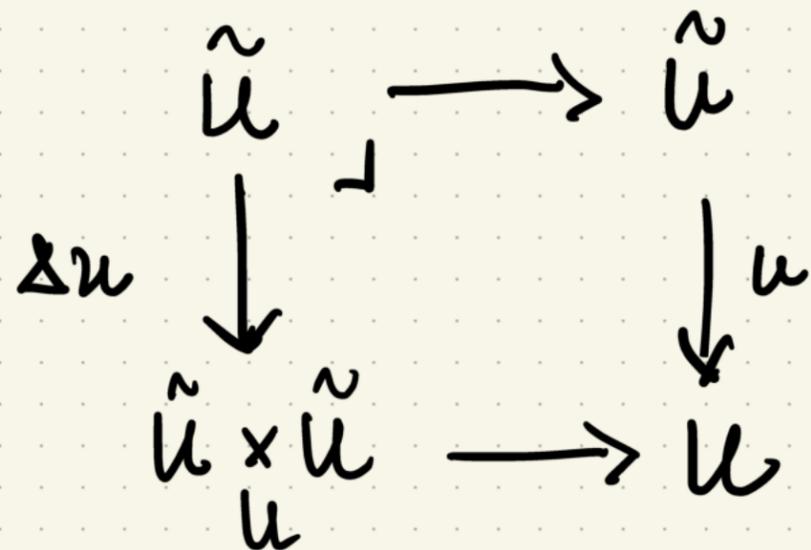
Work in Progress

• Π -types

$$u = t + u \times u$$

• Eq

$$\Delta u \rightarrow u$$



• Id

...

Thanks Thierry,

for 10⁺ years

of inspiration!