Investigating the Tarpeian rock : Paradoxes in Type Theory

Thanks to Thierry for explanation and inspiration!

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All attemps to strengthen this system the system of constructions in particular to temp er with the fourth level should be very cautious: the Tarpeian Rock is close to the Capitol

J.Y. Girard

The paradox of trees

Coquand 92

The paradox of trees in type theory.
BIT Numerical Mathematics. 1992 Mar;32(1):10-4.

- Russell paradox in Type Theory with Type : Type
- Sets as trees

Using Type:Type

```
data Tree : Set where
  node : (I : Set) → (I → Tree) → Tree
```

```
[] : Tree
[] = node ⊥ λ ()
[[]] : Tree
[[]] = node ⊤ (λ _ → [])
[[],[[]]] : Tree
[[],[[]]] = node Bool (λ x → if x then [] else [[]])
```

Derving the paradox

```
data _∈_ : Tree → Tree → Set where
  ∈-in : {I : Set} (f : I → Tree) (i : I) → f i ∈ node I f

Good : Tree → Set
Good t = ¬ (t ∈ t)

Russell : Tree
Russell = node (Σ[ t ∈ Tree ] Good t) proj₀

strange : Good Russell ⇔ ¬ (Good Russell)
proj₀ strange h = ∈-in proj₀ (Russell , h)
proj₁ strange (∈-in ∘ (proj₀) (.Russell , good)) = good

ex : ¬ (¬ P ⇔ P)
ex = ?
```

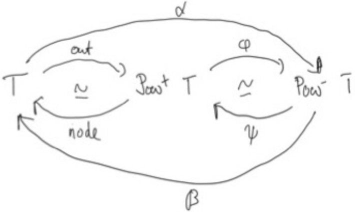
Analyzing the tree paradox

Pow- , Pow+ : Set $\rightarrow$ Set

Pow- A = A $\rightarrow$ Set
Pow+ A = Σ[I $\in$ Set] (I $\rightarrow$ A)
(α , β) : Pow+ A $\simeq$ Pow- A
(node, out) : Pow+ Tree $\simeq$ Tree

Cantor: The following is inconsistent:

α : X $\rightarrow$ Pow- X β : Pow- X $\rightarrow$ X
ret : (P : Pow- X) (x : X) $\rightarrow$ α (β P) X $\Leftrightarrow$ P X



Are we done?

- The paradox of trees is intuitively clear and has a short formalisation.
- But is it as strong as Girard's paradox?
- Girard's paradox doesn't require Type:Type and only need Π-types.

Higher order logic (HOL)

- In Higher Order Logic we can quantify over propositions.
- As an example we can encode inductive definitions, that is given $A : \mathsf{Set}$ and $\_<\_ : A \to A \to \mathsf{Prop}$ we can replace the inductive definition

$$
\begin{array}{l}
\textbf{data}\ \mathsf{Acc} : A \to \mathsf{Prop}\ \textbf{where} \\
\quad \mathsf{acc} : \{a : A\} \to (\{x : A\} \to x < a \to \mathsf{Acc}\ x) \to \mathsf{Acc}\ a
\end{array}
$$

by

$$
\begin{array}{l}
\mathsf{Acc} : A \to \mathsf{Prop} \\
\mathsf{Acc}\ x = (P : A \to \mathsf{Prop}) \\
\quad \to ((a : A) \to (\{b : A\} \to b < a \to P\ b) \to P\ a) \\
\quad \to P\ x
\end{array}
$$

- Gödel introduced System T as a functional calculus corresponding to Heyting Arithmetic (HA)
- Similarly Girard introduced System F as a functional calculus corresponding to Higher Order Logic.

$$\frac{\text{System T}}{\text{HA}} = \frac{\text{System F}}{\text{HOL}}$$

- $\text{HA}^\omega = \text{HA} + \text{System T}$
- $\text{SystemU} = \text{HOL} + \text{System F}$
- Alas, System U is inconsistent.

System U in Agda

```
Type = Set₁
Prp = Set₀
record Pi {ℓ} (A : Set ℓ) (B : A → Type) : Type where
  constructor lam
  field
    _$_ : (a : A) → B a
record All {ℓ} (A : Set ℓ) (B : A → Prp)  : Prp where
  constructor lam-p
  field
    _$p_ : (a : A) → B a
```

Using `NO_UNIVERSE_CHECK`.

Girard's paradox

Coquand 86

An analysis of Girard's paradox LICS 86

- Girard uses the Burali-Forti paradox about the well-order of all well orderings.
- In System U we can define a universal System to represent relations:

$$
\begin{aligned}
&\mathrm{U} \;:\; \mathrm{Type} \\
&\mathrm{U} \;=\; (\Pi[\, \mathrm{B} \in \mathrm{Type} \,]\, ((\mathrm{B} \;\rightarrow\; \mathrm{B} \;\rightarrow\; \mathrm{Prp}) \;\rightarrow\; \mathrm{Prp})) \;\rightarrow\; \mathrm{Prp} \\
&\text{in-U} \;:\; (\mathrm{A} \;:\; \mathrm{Type})\, (\mathrm{R} \;:\; \mathrm{A} \;\rightarrow\; \mathrm{A} \;\rightarrow\; \mathrm{Prp}) \;\rightarrow\; \mathrm{U} \\
&\text{in-U A R f} \;=\; (\mathrm{f} \;\$\; \mathrm{A})\, \mathrm{R}
\end{aligned}
$$

- We can define the order on well-orders on U.
- It is important that in-U A R $\equiv$ in-U B R implies that there is a monotone function from A , R to B , R

- We can also construct the Burali-Forti paradox on T defining the tree of all well-founded trees.
- Can we just use irreflexive relations instead of well-founded ones for Girard's paradox?

Girard's paradox ?

- A complete formalisation of Girard's paradox is quite complex.
- Can we avoid using Burali-Forti?
- Can we use an impredicative inductive definition to simplify the paradox?

A new paradox?

Coquand 95

A new paradox in type theory Studies in Logic and the Foundations of Mathematics. Vol. 13 86

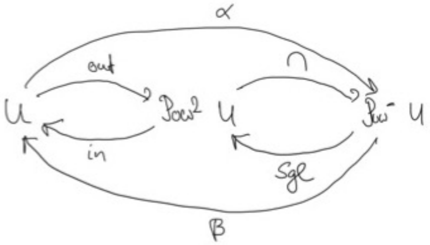
Reynolds 84

Polymorphism is not set-theoretic International Symposium on Semantics of Data Type

- This paradox is based on Reynold's observation that there is no set theoretic semantics of System F.
- We can encode a fixpoint of $\mathsf{Pow}^2\ \mathsf{A}\ =\ \mathsf{Pow\text{-}}\ (\mathsf{Pow\text{-}}\ \mathsf{A})$

$$
\begin{aligned}
&\mathsf{U}\ :\ \mathsf{Set}\\
&\mathsf{U}\ =\ (\mathsf{X}\ :\ \mathsf{Set})\ \to\ (\mathsf{Pow}^2\ \mathsf{X}\ \to\ \mathsf{X})\ \to\ \mathsf{X}\\
&\mathsf{inn}\ :\ \mathsf{Pow}^2\ \mathsf{U}\ \to\ \mathsf{U}\\
&\mathsf{out}\ :\ \mathsf{U}\ \to\ \mathsf{Pow}^2\ \mathsf{U}
\end{aligned}
$$

$\bigcap \ : \ \text{Pow}^2 \ U \ \rightarrow \ \text{Pow-} \ U$
$\bigcap Q\,x \ = \ (P \ : \ \text{Pow-} \ U) \ \rightarrow \ Q\,P \ \rightarrow \ P\,x$
$\text{Sgl} \ : \ \text{Pow-} \ U \ \rightarrow \ \text{Pow}^2 \ U$
$\text{Sgl}\,P \ = \ \lambda\,Q \ \rightarrow \ (x \ : \ U) \ \rightarrow \ P\,x \Leftrightarrow Q\,x$
$\cap\,(\text{Sgl}\,P)\,X \Leftrightarrow P\,X$



Partial Equivalence Relations (PERs)

- We need that out (inn x) $\equiv$ x !
- But the impredicative encoding only gives us
  Pow^2 (inn $\circ$ out) $\equiv$ out $\circ$ inn
- This can be addressed by interpreting the construction in the PER model.
- We define (U,Eq) using an impredicative definition.

$$\begin{aligned}\mathsf{Eq}\ x\ y\ =\ (R\ :\ U\ \to\ U\ \to\ \mathsf{Set})\ \to\ \mathsf{per}\ R \\ \to\ \mathsf{mor}\ (\mathsf{pow}^2 R\ R)\ R\ \mathsf{inn}\ \to\ R\ x\ y\end{aligned}$$

Hurken's paradox

Hurken 95

A Simplification of Girard's paradox TLCA 95

- Hurken uses the same U as Coquand.
- He avoids the use of PERs!
- He encodes a variant of the Burali-Forti paradox.

```
Ind : Pow² U
Ind X = ∀[ x ∈ U ] (out x X → X x)

Acc : Pow U
Acc x = ∀[ X ∈ Pow U ] (Ind X → X x)

Ω : U
Ω = inn Ind

accΩ : Acc Ω
accΩ = λp[ X ] λ iX → (iX $p Ω) (λp[ y ] (iX $p (inn (out y))))

_<_ : U → Pow U
x < y = ∀[ X ∈ Pow U ] (out y X → X x)

¬Δ : Pow U
¬Δ y = ¬ (inn (out y) < y)

lem : Ind ¬Δ
lem = λp[ x ] λ h p → (p $p ¬Δ) h (λp[ X ] λ q → (p $p λ y → X (inn (ou

¬accΩ : ¬ (Acc Ω)
¬accΩ h = (h $p ¬Δ) lem (λp[ X ] (λ p → (h $p λ y → X (inn (out y))) p))

paradox : ⊥
paradox = ¬accΩ accΩ
```

Comparing paradoxes

The paradox of trees Short and easy to understand, but needs Type:Type and W.

Girard's paradox We need to formalize Burali-Forti!

Coquand's paradox We need to develop the PER machinery.

Hurken's paradox Nice and short but hard to explain.