There are more groups in the next universe

Martín Escardó
University of Birmingham, UK

j.w.w. Marc Bezem
Thierry Coquand
Peter Dybjer

Workshop in honour of Thierry Coquand's 60th
24-26 August 2022
Gothenburg, Sweden

Personal interactions with Thierry

- I remember vividly lots of conversations at Domains' 2002 in Birmingham.
- And in all Formal Topology worshops except the first one.
- And in a number of Dagstuhl Seminars.
- They were never about type theory.
- Always about constructive mathematics and pointfree topology.
- Then in approximatly 2010 I got interested in type theory & Agda.
- In 2012 I suggested to invite Thierry to explain HoTT/UF at MGS.
- Since then we never stopped discussing univalent type theory.

The work presented here

- In 2019, Marc Bezem & Bjørn Dundas organized a research programme Homotopy Theory and Univalent Foundations in OSLO at the Centre for Advanced Study.

- Marc, Thierry, Peter and I wanted to understand universes from two points of view:

  1. Metamathematical (cf. Peter's talk at Types'22)
  2. Mathematical (this talk)

A recurrent theme: type theory with $\mathcal{U} : \mathcal{U}$ is inconsistent

metatheorem

- A better view.

In type theory without such an outrageous judgment, one can show that there is no type $X : \mathcal{U}$ isomorphic to $\mathcal{U}$.

theorem

- Mathematical essence of the situation:

1. $\mathcal{U} : \mathcal{U}^+$ (the universe $\mathcal{U}$ lives in the next universe $\mathcal{U}^+$)
2. We have an embedding $\mathcal{U} \to \mathcal{U}^+$.
3. There are types $Y : \mathcal{U}^+$ with no isomorphic copy $X : \mathcal{U}$.

There are more types in the next universe.

It should be the case that

- There are more sets in the next universe. — in the sense of HoTT/UF.

  use Burali-Forti

- More magmas.
- More monoids.
- More groups. — harder
- More spaces.
- More categories.
- etc.

addressed here.

not addressed here
(and we didn't think about them)

Let's fix our type theory first

- $\Pi, \Sigma, +, \mathbb{0}, \mathbb{1}, \mathbb{N}, \underbrace{\mathcal{U}_0, \mathcal{U}_1, \mathcal{U}_2, \ldots}$
  (no W types)

  universes, not assumed to be (or not to be) cumulative.

  use Agda-like treatment of universes (cf. Peter's Types'22 talk)

- Univalence
  (hence functional and propositional extensionality)

- Propositional truncation $\|-\|$
  (the only HIT)

This is a kind of Spartan HoTT/UF I adopt in my Agda github repository TypeTopology.

Getting started

This relies on univalence.

1. There is a unique map $e : \mathcal{U} \to \mathcal{U}^+$ with $eX \simeq X$ for all $X : \mathcal{U}$.
2. This map is an $\underbrace{\text{embedding}}_{\substack{\text{all fibers are} \\ \text{subsingletons} \\ \text{(or propositions)}}}$, but $\underbrace{\text{not an equivalence}}_{\substack{\text{not all fibers are} \\ \text{singletons}}}$.
3. There is a specific $Y : \mathcal{U}^+$ such that its fiber

$$e^{-1} Y := \Sigma\, X : \mathcal{U},\ eX = Y$$

identity type

is empty.

This says that there are (strictly) more types in the next universe.

Proof ideas

1. The type $\mathcal{O}$ of ordinals in $\mathcal{U}$ is an ordinal in the next universe $\mathcal{U}^+$.
   - The Burali-Forti argument shows that this ordinal $\mathcal{O}$ is not isomorphic to any ordinal in $\mathcal{U}$.
   - If the map $e : \mathcal{U} \to \mathcal{U}^+$ were an equivalence, then $\mathcal{O}$ would be isomorphic to an ordinal in $\mathcal{U}$.

So $\mathcal{O}$ is an example of a type in $\mathcal{U}^+$ with no isomorphic copy in $\mathcal{U}$.

Proof ideas ctd.

2. The type $\mathbb{O}$ is a set (by univalence)

for $\alpha, \beta : \mathbb{O}$, the identity type $\alpha = \beta$ is a subsingleton.

and can be given the structure of magma and of a monoid using ordinal addition and the ordinal zero.

- But we don't think it can be given the structure of a group constructively. (open question)
- Idea: consider the group freely generated by $\mathbb{O}$.

Proof ideas ctd.

- Idea: consider the group freely generated by $\mathbb{O}$.
  - There are some technical issues putting this idea into practice.
  - In particular, because we are not including HITs other than $\|-\|$.
  - So we can't construct free groups and quotients using HITs.
- I'll give an indication of how we overcome these obstacles.
- But first we need to develop some background on ordinals.

Ordinals in univalent type theory

An ordinal in a universe $\mathcal{U}$ is a type $X : \mathcal{U}$ equipped with a relation $\_ < \_ : X \to X \to \mathcal{U}$ required to be

1. proposition valued,
2. transitive
3. extensional (any two points with the same lower set are themselves the same)
4. well founded (every element is accessible, or, equivalently, the principle of transfinite induction holds)

The HoTT book additionally requires the type X to be a set, but this follows automatically from (3), by Hedbergs theorem.
(E.g. the type $\mathbb{N}$ of natural numbers, and the types $\mathrm{Fin}\, n$ are ordinals.)

Equivalence and equality of ordinals

Def.

Equivalence of ordinals in universes $\mathcal{U}$ and $\mathcal{V}$,

$$- \simeq_o - : \text{Ordinal}\ \mathcal{U} \to \text{Ordinal}\ \mathcal{V} \to \mathcal{U} \sqcup \mathcal{V},$$

means that that there is an equivalence of the underlying types that preserves and reflects order.

Fact.

For ordinals α, β in the same universe $\mathcal{U}$, there is a canonical bijection $\underbrace{\alpha = \beta}_{\text{this type lives in } \mathcal{U}^+} \to \underbrace{\alpha \simeq_o \beta}_{\text{this type lives in } \mathcal{U}}$

Important!

(So univalence gives some amount of resizing down.)

Some facts about ordinals

1. The lower set $\alpha \downarrow x$ of a point $x : \langle \alpha \rangle$ of the underlying type $\langle \alpha \rangle$ of an ordinal α is itself an ordinal (under the inherited order).

2. The ordinals in a universe $\mathcal{U}$ form an ordinal in the next universe $\mathcal{U}^+$ with

   $$\alpha \lhd \beta := \sum b : \langle \beta \rangle ,\ \alpha = \beta \downarrow b .$$

   This order has type

   $$\text{Ordinal}\ \mathcal{U} \to \text{Ordinal}\ \mathcal{U} \to \mathcal{U}^+$$

   as required to make the type $\text{Ordinal}\ \mathcal{U}$ into an ordinal in $\mathcal{U}^+$.

$\boxed{\text{Facts about ordinals ctd.}}$

The above compares ordinals in the same universe.

- It is possible, and crucial for our purposes, to compare ordinals in different universes.
- We define

$$\_ \lhd^- \_ : \mathrm{ordinal}\ \mathcal{U} \to \mathrm{ordinal}\ \mathcal{V} \to \mathrm{ordinal}\,(\mathcal{U} \sqcup \mathcal{V})$$

by

$$\alpha \lhd^- \beta := \Sigma\, b : \langle \beta \rangle,\ \alpha \simeq_o \beta \downarrow b$$

- When the universes $\mathcal{U}$ and $\mathcal{V}$ are the same, this resizes

$$\_ \lhd \_ : \mathrm{ordinal}\ \mathcal{U} \to \mathrm{ordinal}\ \mathcal{U} \to \mathcal{U}^+$$

down to

$$\_ \lhd^- \_ : \mathrm{ordinal}\ \mathcal{U} \to \mathrm{ordinal}\ \mathcal{U} \to \mathcal{U}$$

Facts about ordinals ctd.

- The existence of such a resized-down order is crucial for the corollaries of Burali-Forti, but not for Burali-Forti itself.

- We denote by $\mathrm{O}\mathcal{U} : \mathrm{Ordinal}\, \mathcal{U}^{+}$ the ordinal of ordinals in $\mathcal{U}$, that is,

$$\mathrm{O}\mathcal{U} := (\mathrm{ordinal}\, \mathcal{U}, - \triangleleft -).$$

- For any $\alpha : \mathrm{ordinal}\, \mathcal{U}$, we have
$$\alpha \simeq_o \mathrm{O}\mathcal{U} \downarrow \alpha$$
and hence
$$\alpha \triangleleft^{-} \mathrm{O}\mathcal{U}.$$

Burali-Forti in univalent type theory

Theorem. No ordinal in a universe $\mathcal{U}$ can be equivalent to the ordinal of all ordinals in $\mathcal{U}$.

Proof. Suppose, for the sake of contradiction, that there is

$$\alpha \simeq_o \mathcal{O}\mathcal{O}\mathcal{U}.$$

Because

$$\alpha \simeq_o \mathcal{O}\mathcal{O}\mathcal{U} \downarrow \alpha,$$

we get

$$\mathcal{O}\mathcal{O}\mathcal{U} \simeq_o \mathcal{O}\mathcal{O}\mathcal{U} \downarrow \alpha$$

and so

$$\mathcal{O}\mathcal{O}\mathcal{U} \triangleleft \mathcal{O}\mathcal{O}\mathcal{U},$$

which is impossible as any accessible relation is irreflexive. ∎

corollary

Theorem. There are more sets, and hence more types, in the next universe.

Proof. The type $\mathrm{Ord}\,\mathcal{U} : \mathcal{U}^+$ is a set by univalence, as

$$(\alpha = \beta) \simeq (\alpha \simeq_o \beta)$$

and the type $\alpha \simeq_o \beta$ has at most one element (two ordinals are isomorphic in at most one way) ∎

$\boxed{\text{Locally small type}}$ (Egbert Rijke)

A type $X : \mathcal{U}^+$ is locally small if for every $x, y : X$, the identity type

$$x = y : \mathcal{U}^+$$

is isomorphic to some designated type in $\mathcal{U}$.

Example. The type of ordinals in $\mathcal{U}$ is large but locally small.

Quotients in our type theory

An equivalence relation on $X : \mathcal{U}$ is a relation $\approx : X \to (X \to \mathcal{U})$ which is proposition-valued, reflexive, symmetric and transitive.

The 'image' of $\approx$, namely

$$X/\approx \;:=\; \Sigma\, \varphi : X \to \mathcal{U},\ \left\| \Sigma\, x : X,\ \approx(x) = \varphi \right\|$$

has the universal property of a quotient

transform $\approx$ into $=$

$$\begin{tikzcd} X \arrow[r, "\eta"] \arrow[dr, "f"'] & X/\approx \arrow[d, dashed, "\bar{f}"] \\ & A \end{tikzcd}$$

Moreover, the quotient is effective.
(η reflects $=$ into $\approx$)

$\boxed{\text{However}}$

$X/\!\approx$ constructed in this way lives in the next universe.

We'll have to deal with this when we construct free groups.

Free groups

Based on "A course on constructive algebra"
by Mines, Richman & Ruitenberg, 1988.

- Given a set A of generators, for $(n, a) : 2 \times A$ define
  $(n, a)^- :=$ (complement n, a)
- We quotient $\mathrm{List}(2 \times A)$ by a suitable equivalence relation $\approx$ to get the group freely generated by A.
- This explicit construction allows us to see that the insertion of generators $\eta : A \to FA$ is one-to-one, which is crucial for our purposes.

Free group ctd.

- Define $s \rhd t$, for $s, t : \mathrm{List}(2 \times A)$, to be the type

$$\sum \mu, \nu : \mathrm{List}(2 \times A), \sum x : 2 \times A, \ (s = \mu ++ [x] ++ [x^-] ++ \nu) \times (t = \mu ++ \nu)$$

Lemma (Church–Rosser property, MRR, 88)

$$\mu_0 ++ [x_0] ++ [x_0^-] ++ \nu_0 = \mu_1 ++ [x_1] ++ [x_1^-] ++ \nu_1$$

$$\to (\mu_0 ++ \nu_0 = \mu_1 ++ \nu_1) + (\Sigma t, (\mu_0 ++ \nu_0 \rhd t) \times (\mu_1 ++ \nu_1 \rhd t))$$

Corollary

$A \triangleright t_0$ and $A \triangleright t_1$

together imply

$$(t_0 = t_1) + \left(\Sigma\, t,\ (t_0 \triangleright t) \times (t_1 \triangleright t)\right)$$

It is remarkable that the above doesn't require A to have decidable equality.

Free group ctd.

The group freely generated by A is given by $A/\approx$ where $\approx$ is the propositional truncation of the reflexive, symmetric, transitive closure of $\triangleright$.

- Using our quotient construction, if

$$A : \mathcal{U}$$

then

$$FA : \mathcal{U}^{+}.$$

- So $F(\infty\mathcal{U}) : \mathcal{U}^{++}$. But we need an example in $\mathcal{U}^{+}$.

Free group of a largely, locally small set

Lemma. If $A : \mathcal{U}^+$ is locally small, then $FA : \mathcal{U}^{++}$ has an isomorphic copy in $\mathcal{U}^+$.

Proof idea. The relation $\triangleright$ has type $\mathrm{List}(2 \times A) \to \mathrm{List}(2 \times A) \to \mathcal{U}^+$.
Using the local smallness of A, we can resize it down to have values in $\mathcal{U}$. ∎

Resizing $\rhd$ down to $\blacktriangleright$ (sketch)

$$\_ \mathbin{◗} \_ \,,\; \_ \blacktriangleright \_ \;:\; \mathrm{List}\,(2 \times A) \to \mathrm{List}\,(2 \times A) \to \mathcal{U}$$

$$\begin{cases} [\,] \mathbin{◗} t = \mathbb{0} \\ (x :: [\,]) \mathbin{◗} t = \mathbb{0} \\ (x :: y :: s) \mathbin{◗} t = (y = x^{-}) \times (s = t) \end{cases}$$

$$\begin{cases} [\,] \blacktriangleright t = \mathbb{0} \\ (x :: s) \blacktriangleright [\,] = (x :: s) \mathbin{◗} [\,] \\ (x :: s) \blacktriangleright (y :: t) = \big((x :: s \mathbin{◗} (y :: t)\big) + \big((x = y) \times (s \blacktriangleright t)\big) \end{cases}$$

The usual definition of s.r.t. closure of $\triangleright$ would increase again the universe levels because A is large

$$
\begin{aligned}
&\mathrm{redex} : \mathrm{List}\,(2 \times A) \to \mathcal{U} \\
&\mathrm{redex}\ [\,] = \mathbb{0} \\
&\mathrm{redex}\ (x :: [\,]) = \mathbb{0} \\
&\mathrm{redex}\ (x :: y :: s) = (y = x^{-}) + \mathrm{redex}\,(y :: s)
\end{aligned}
$$

$$
\begin{aligned}
&\mathrm{reduct} : (s : \mathrm{List}\,(2 \times A)) \to \mathrm{redex}\ s \to \mathrm{List}\,(2 \times A) \\
&\mathrm{reduct}\,(x :: y :: s)\ (\mathrm{inl}\ p) = s \\
&\mathrm{reduct}\,(x :: y :: s)\ (\mathrm{inr}\ r) = x :: \mathrm{reduct}\,(y :: s)\ r
\end{aligned}
$$

$\boxed{\text{Resizing } F(\infty\mathcal{U}) : \mathcal{U}^{++} \text{ down to } \mathcal{U}^{+} \text{ ctd}}$

with the above we can finally resize $\approx$ to have values in $\mathcal{U}$ rather than $\mathcal{U}^+$,

and hence $F(\infty\mathcal{U})$ to live in $\mathcal{U}^+$ rather than $\mathcal{U}^{++}$.

One final step is needed

Lemma. If FA is small, then A is small.

Corollary. $F(\text{OOU})$ is large

This means that there is a group in $\mathcal{U}^+$ with no isomorphic copy in $\mathcal{U}$.

small things

1. $X : \mathcal{U}^+$ is small if it is isomorphic to some $Y : \mathcal{U}$.
2. $X : \mathcal{U}^+$ is locally small if $x = y$ is small for all $x, y : X$.
3. $f : X \to Y$ is a small map if its fibers are small.

$X : \mathcal{U}^+$

Lemma (Tom De Jong & E.)
If f is small and Y is small, then X is small.

Lemma The insersion of generators $\eta : A \to FA$ is a small map if A is (large but) locally small.